

Contributor Profiles: Richard K. Guy



Richard's connection to *Crux Mathematicorum* has been invisible to most readers. But for the nine-year period 1986–1995, while Bill Sands was Editor-in-Chief, they shared an editorial office at the University of Calgary and discussed a fair proportion of the problems published in *Crux Mathematicorum* during that period. From 1991 to 2003, Richard was also on the

Editorial Board of CRUX. This overlapped Richard's own 25-year period of service as Unsolved Problems Editor of the *American Mathematical Monthly*. Richard also founded the *Skoliad*, and continues an occasional collaboration with the Olympiad Corner Editor, colleague Robert Woodrow. Readers may have noticed a short article by Richard printed in the February 2007 issue [2007 : 37–39].

He has published 300 papers; he has edited *Reviews in Number Theory* (1973–1983) and *Proc. Symp. Appl. Math.* 43(1991) for the AMS; he has collaborated with John Conway and Elwyn Berlekamp in writing the four volumes of *Winning Ways*, with Hallard Croft and Ken Falconer in *Unsolved Problems in Geometry*, with John Conway in *The Book of Numbers*, and with Loren Larsen and Paul Vaderlind in *The Inquisitive Problem Solver*; and he wrote *Unsolved Problems in Number Theory* and *Fair Game* on his own.

When not doing mathematics, he still likes to wander or ski in the mountains, usually with his wife Louise. The picture shows him on the summit of The Towers on his 90th birthday in September 2006, with Mt. Assiniboine in the background.

According to Richard, he is the luckiest Guy in the world! There are many reasons for such a claim, the most important ones being

(i) he has been married to the best wife imaginable for 66 years; together they raised 3 children, and they currently have 5 grandchildren and 2 great grandchildren;

(ii) he was paid for doing what he liked—doing mathematics and telling other people about it—for 43 years;

(iii) for the last 25 years, he has continued to be honoured and rewarded for doing the mathematics he likes to do; and

(iv) he has been privileged to know and work with some of the best mathematicians in the world: Elwyn Berlekamp, John Conway, Pál Erdős, the Lehmers, Eric Milner, Alexander Oppenheim, John Selfridge,

SKOLIAD No. 100

Robert Bilinski

Please send your solutions to the problems in this edition by **1 September, 2007**. A copy of **MATHEMATICAL MAYHEM Vol. 2** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

Nos questions proviennent ce mois-ci du Concours de l'Association Mathématique du Québec 2005 (niveau secondaire). Nous remercions Véronique Hussin, Université de Montréal qui s'occupe des concours de l'AMQ du secondaire.

Concours de l'Association Mathématique du Québec (niveau secondaire) 3 février 2005

1. (Le robot et les pommes.) Une caisse de bois est séparée en 9 compartiments comme indiqué sur le dessin. Un ingénieur a programmé un robot pour qu'il remplisse la caisse de pommes par paquets de quatre en laissant tomber une pomme dans chaque compartiment de façon à former un carré 2×2 .

| | | |
|----|----|----|
| 6 | 16 | 10 |
| 11 | 28 | 17 |
| 5 | 12 | 7 |

Est-il possible pour le robot d'aboutir à la configuration à droite à partir d'une caisse vide ?

2. (Huit carrés dans un rectangle.) Diviser un rectangle de longueur égale à 9 cm et de largeur égale à 3 cm en huit carrés.

3. (Une étonnante distribution.) Une distribution statistique est composée de 10 nombres naturels : $x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5$. Lorsqu'ils sont placés en ordre croissant, ces nombres nous donnent en fait la distribution suivante : $x_1, x_2, x_3, x_4, x_5, y_5, y_4, y_3, y_2, y_1$. Nous avons plusieurs informations :

- (1) Les couples $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ et (x_5, y_5) , sont tous sur la droite d d'équation $y = -2x + 24$.
- (2) La moyenne de cette distribution est 9,4.
- (3) La médiane et le mode ont tous deux la même valeur.
- (4) Les nombres x_3 et x_4 sont consécutifs.
- (5) Le premier membre de la distribution vaut 1.
- (6) La droite d croise la parabole d'équation $y = \frac{1}{2}x^2 + 8x - 8$ au point (x_2, y_2) .

Trouver les valeurs de la distribution originale $x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5$. Suggestion : La médiane est le nombre tel que 50% des observations sont plus petites ou égales à ce nombre et 50% supérieures ou égales. Le mode est la valeur qui est observée le plus souvent.

4. (La belle somme de Gilbert.) Considérons les 6 façons possibles de permuter (c'est-à-dire mélanger) les chiffres du nombre 123 et additionnons le tout. La somme trouvée s'écrit $123 + 132 + 213 + 231 + 312 + 321 = 1332$.

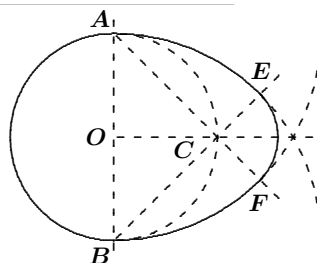
Quel résultat aurions-nous obtenu si nous avions fait la somme des 5040 façons de permuter les chiffres du nombre 1234567 ?

5. (Le voyage à Québec.) Juliette et Philippe partent en même temps et parcourent les 250 km qui séparent Montréal de Québec dans deux voitures identiques. Philippe parcourt la première moitié du trajet à 80 km/h et la seconde moitié à 120 km/h. En fait, il arrive en même temps que Juliette qui a roulé tout le long à vitesse constante. La consommation d'essence de ce type de voiture dépend de la vitesse du véhicule. Elle est donnée par la formule $c = 10 + \frac{v}{20}$, où v est la vitesse en km/h et c la consommation en litres par 100 km. Sachant que ce jour-là, le litre d'essence vaut 0,80\$, combien ont-ils dépensé ensemble pour le voyage ?

6. (Les âges multiples.) Du 21 août 1989 au 7 mai (inclusivement) 1990, Jean a eu 5 fois l'âge de sa fille Claire. Du 8 mai au 20 août (inclusivement) 1992, Jean a eu 4 fois l'âge de sa fille. Trouver la date de naissance de chacun.

Note : par âge, on entend la définition usuelle qui est le nombre d'années complètes écoulées depuis la naissance.

7. (La poule géomètre.) Une figure plane en forme d'œuf est délimitée par quatre arcs de cercles désignés par \widehat{AB} , \widehat{BF} , \widehat{FE} et \widehat{EA} mis bout à bout de la façon indiquée par la figure à droite. Sachant que le rayon AO est de longueur 1, déterminer l'aire de la figure.



Contest of the Mathematical Association of Quebec (Secondary Level) February 3, 2005

1. (The robot and the apples.) A wooden case is separated into 9 compartments as in the drawing. An engineer has programmed a robot to fill the case with apples by dropping four apples at a time, one into each compartment of a 2×2 square.

Is it possible for the robot to finish with the configuration shown if it starts with an empty case?

| | | |
|----|----|----|
| 6 | 16 | 10 |
| 11 | 28 | 17 |
| 5 | 12 | 7 |

2. (Eight squares in a rectangle.) Divide a rectangle of length 9 cm and width 3 cm into eight squares.

3. (An astonishing distribution.) A statistical distribution is composed of 10 natural numbers: $x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5$. When placed in increasing order, these numbers are: $x_1, x_2, x_3, x_4, x_5, y_5, y_4, y_3, y_2, y_1$. We also have the following information:

- (1) The couples $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$, and (x_5, y_5) are all on the line d with equation $y = -2x + 24$.
- (2) The mean of the distribution is 9.4.
- (3) The median and the mode are both the same value.
- (4) The numbers x_3 and x_4 are consecutive.
- (5) The first member of the distribution is 1.
- (6) The line d crosses the parabola having equation $y = \frac{1}{2}x^2 + 8x - 8$ at the point (x_2, y_2) .

Find the values of the original distribution $x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5$. Note: The median is the number such that 50% of the observations are less than or equal to the number and 50% are greater than or equal. The mode is the value which is repeated most often.

4. (Gilbert's beautiful sum.) Consider the 6 different numbers obtained by permuting (mixing) the digits of the number 123. The sum of these numbers is $123 + 132 + 213 + 231 + 312 + 321 = 1332$.

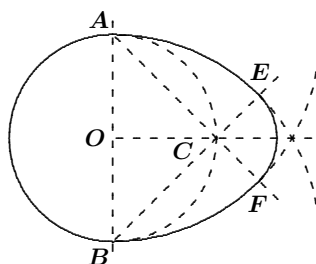
What result would be found if we summed the 5040 different numbers obtained by permuting the digits of the number 1234567?

5. (The trip to Quebec.) Julia and Phillip leave at the same time and cross the 250 km that separate Montreal and Quebec in two identical cars. Phillip does the first half of the trip at 80 km/h and the second half at 120 km/h. He arrives at the same time as Julia, who travelled at a constant speed the whole trip. The fuel consumption for that type of car depends on the speed of the vehicle. It is given by the formula $c = 10 + \frac{v}{20}$, where v is the speed in km/h and c the consumption in liters per 100 km. On that particular day, one litre of gasoline cost \$0.80. How much did they spend on gas altogether for the trip?

6. (Age multiples.) From August 21, 1989 to May 7, 1990 (inclusive), John was 5 times as old as his daughter Claire. From May 8, 1992 to August 20, 1992 (inclusive), John was 4 times as old as his daughter. Find the date of birth of each of them.

Note: By age, we mean, as usual, the number of complete years that have passed since birth.

7. (The geometric hen.) An egg-shaped figure in the plane is composed of four arcs of circles, designated by \widehat{AB} , \widehat{BF} , \widehat{FE} , and \widehat{EA} , put end to end as indicated in the figure. Knowing that the radius AO has length 1, determine the area of the figure.



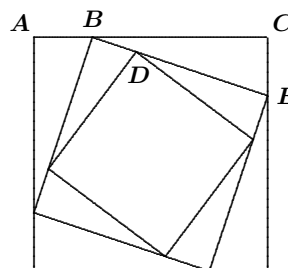
Next we give the solutions to the team round of the fifth annual CNU contest for high school students, run by Ron Persky at Christopher Newport University [2006 : 258–260].

1. Mr. Smith pours a full cup of coffee and drinks $\frac{1}{2}$ of it, deciding it is too strong and needs some milk. So he fills the cup with milk, stirs it, and tastes again, drinking another $\frac{1}{4}$ cup. Once again he fills the cup with milk, stirs it, and finds that this is just as he likes it. What ratio $\frac{\text{amount of coffee}}{\text{amount of milk}}$ does Mr. Smith like?

Solution by Anna Beaudin, Montreal, PQ.

After he drinks another $\frac{1}{4}$ cup, only half of the $\frac{3}{4}$ cup that remains is coffee. This means that $\frac{3}{8}$ is coffee, the rest being milk. Hence, the ratio of coffee to milk is $\frac{3}{5}$.

2. You have three inscribed squares, with the corners of each inner square at the $\frac{1}{4}$ point along the sides of its outer square. (Thus, for example, $AB = \frac{1}{4}AC$ and $BD = \frac{1}{4}BE$.) The area of the largest square is 64 cm^2 . What is the area of the smallest square?



Solution by the editor.

By the Theorem of Pythagoras, we have $BC^2 + CE^2 = BE^2$. Thus, $BE^2 = \frac{9}{16}AC^2 + \frac{1}{16}AC^2 = \frac{5}{8}AC^2$. This means that at each stage, the area of the next smaller square is $\frac{5}{8}$ times the area of the current square. Therefore, the area of the smallest square is $(\frac{5}{8})^2 \cdot 64 = 25 \text{ m}^2$.

3. Solve the equation $\cos 2x = \cos x$ for $0 \leq x < 2\pi$.

Solution by the editor.

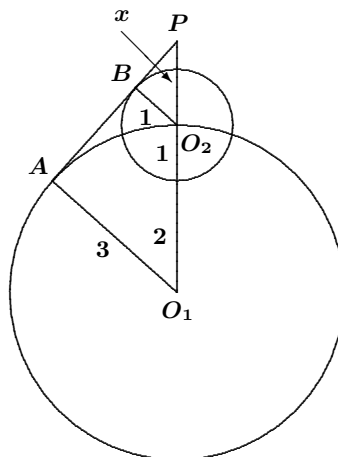
Since $\cos 2x = 2\cos^2 x - 1$, the given condition is equivalent to $2\cos^2 x - \cos x - 1 = 0$, a quadratic equation in the variable $\cos x$. Factoring

gives $(2 \cos x + 1)(\cos x - 1) = 0$, which yields $\cos x = 1$ or $\cos x = -\frac{1}{2}$. Then, since $0 \leq x < 2\pi$, we have $x \in \{0, \frac{2\pi}{3}, \frac{4\pi}{3}\}$.

4. The centre of a circle of radius 1 cm is on the circumference of a circle of radius 3 cm. How far (in cm) from the centre of the big circle do the common tangents of the two circles meet?

Solution by the editor.

Let O_1 and O_2 be the centres of the circles of radii 3 and 1, respectively. Let P be the point where the common tangents of the two circles meet. Let A and B be the points of contact of one of the common tangents with the circles of radii 3 and 1, respectively, as shown. Let x denote the length of O_2P . By similar triangles, we have $x : 1 = (x + 3) : 3$, which simplifies to $x = \frac{3}{2}$. Therefore, the distance from the centre of the big circle to the point where the common tangents meet is $PO_1 = \frac{9}{2}$.



5. One root of $2hx^2 + (3h - 6)x - 9 = 0$ is the negative of the other. Find the value of h .

Solution by the editor.

Note that $h \neq 0$, since the given equation has only one root, $x = -\frac{3}{2}$, if $h = 0$. Divide the equation by $2h$ to get

$$x^2 + \frac{3h - 6}{2h}x - \frac{9}{2h} = 0.$$

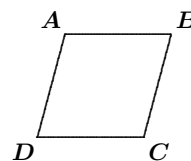
Then the sum of the roots is $-(3h - 6)/(2h)$. However, since one root is the negative of the other, the sum must be 0; that is, $3h - 6 = 0$, which means that $h = 2$.

6. Solve the equation $\sqrt{16x + 1} - 2\sqrt[4]{16x + 1} = 3$.

Solution by the editor.

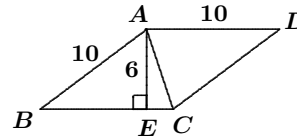
Let $y = \sqrt[4]{16x + 1}$. Then the given equation becomes $y^2 - 2y - 3 = 0$, or $(y - 3)(y + 1) = 0$. Thus, $y = 3$ or $y = -1$. Since $y = \sqrt[4]{16x + 1} > 0$, we cannot have $y = -1$. Therefore, $y = 3$; that is, $\sqrt[4]{16x + 1} = 3$. Then $16x + 1 = 3^4 = 81$, which yields $x = 5$.

7. In the figure $ABCD$, all four sides have length 10 and the area is 60. What is the length of the shorter diagonal AC ?



Solution by the editor.

Drop a perpendicular from the vertex A to the side BC meeting it at E . Since the area of the rhombus is 60, we see that the altitude $AE = 6$. By the Theorem of Pythagoras, we have $BE = 8$, which implies that $EC = 2$. Then the length of the shorter diagonal, AC , can be obtained by applying the Theorem of Pythagoras again:



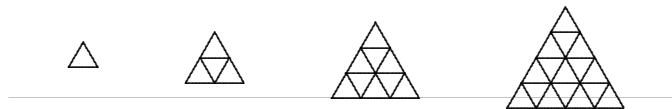
$$AC = \sqrt{6^2 + 2^2} = \sqrt{40} = 2\sqrt{10}.$$

8. A man has 1000 equilateral triangular pieces of mosaic, all of side length 1 cm. He constructs the largest possible mosaic in the form of an equilateral triangle.

- What is the side length of the mosaic?
- How many pieces will he have left over?

Solution by the editor.

(a) Some mosaics in the form of an equilateral triangle that can be constructed from the triangular pieces are shown below.



Notice that the number of pieces in the rows of these mosaics are 1, 3, 5, 7, ... (from top to bottom). If one of these mosaics has sides of length n , then the number of pieces in the bottom row is $2n - 1$, and the total number of pieces in the mosaic is $1 + 3 + 5 + 7 + \dots + (2n - 1) = n^2$. (An alternate approach is to notice that each of these mosaics that can be constructed from the triangular pieces uses n^2 pieces for some positive integer n , and has sides of length n .)

The largest value of n such that $n^2 \leq 1000$ is $n = 31$. Thus, the largest possible mosaic has sides of length 31, which is the answer to part (a). The number of pieces left over is then $1000 - 31^2 = 1000 - 961 = 39$, which answers part (b).

That brings us to the end of another issue. Continue sending in your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as a **Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of *Crux Mathematicorum with Mathematical Mayhem*.

The Mayhem Editor is Jeff Hooper (Acadia University). The Assistant Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are John Grant McLoughlin (University of New Brunswick), Mark Bredin (St. John's-Ravenscourt School, Winnipeg), Monika Khbeis (Father Michael Goetz Secondary School, Mississauga), Eric Robert (Leo Hayes High School, Fredericton), Larry Rice (University of Waterloo), and Ron Lancaster (University of Toronto).

Mayhem Editorial

Jeff Hooper

Welcome to another year of Mathematical Mayhem! I must apologize for my greetings not appearing in the first issue of 2007. Although the first issue of *CRUX with MAYHEM* appears in February of each year, the editing of material for this issue actually occurs in November. With the changeover in duties and the usual rush of late fall, I simply missed doing this.

Before introducing myself as the new editor of Mathematical Mayhem, I wish to thank my predecessor, Shawn Godin. In the middle of last year, Shawn reluctantly decided to step down as Mayhem editor. Shawn is an Ottawa-area high school teacher and education consultant, who has managed to squeeze in all of his Mayhem duties on top of his already heavy schedule. He also recently decided to return to university to work toward his doctorate. With all of these commitments and a young family, Shawn felt that it was the proper time to step down.

It would be difficult to overstate Shawn's contribution to Mayhem. Over his years of involvement, he has kept Mayhem alive and true to its original purpose as a source of mathematical problems and problem-solving ideas suitable for high school students. We will continue down that road. We will miss you, Shawn, and we wish you well with your new challenges! You are welcome back anytime!

Now, a few words about your new Mayhem editor. I am an Associate Professor in the Department of Mathematics and Statistics at Acadia University in Nova Scotia. Before that, I spent several years at the Universities of Cambridge, Durham, and Waterloo. My main area of mathematical interest is number theory, but I also have an interest in mathematics education. This educational slant has led me down a number of other paths, including curriculum consulting for the Nova Scotia Department of Education and serving

as Nova Scotia's provincial coordinator for the Maritime Mathematics Competition written by regional high school students. Many problems from these competitions have appeared previously in the Skoliad. When not busy with all of these other things, I usually spend time with my children, or else play music (poorly).

Last fall the **CRUX with MAYHEM** Editorial Board began a discussion regarding the focus of *Pólya's Paragon*. When *Pólya's Paragon* was started a number of years ago by Paul Ottaway, there was a specific direction to the articles: they focused on a technique or idea that was important in problem-solving and then illustrated that technique on two or three problems. Since then, the content of the Paragon has varied quite a lot. We have decided to return to the original format for future Paragons. This means that we will not have a Paragon every month, but only when we find an article that fits the mold. However, we will continue to welcome articles. I wish to invite readers to submit to us short articles intended for a high school audience. If you feel they fit the mold for a Paragon article, be sure to point that out. We are also always in need of good Mayhem problems. If you can supply us with some, we would be most grateful.

In closing, let me thank all of you for the work you put into Mayhem, either directly or even by just reading and working the problems. I would be very happy to hear from you if you have comments or suggestions. I hope you enjoy the 2007 volume of Mayhem!

Mayhem Problems

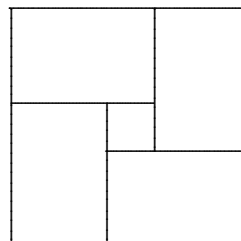
Veillez nous transmettre vos solutions aux problèmes du présent numéro avant le premier juillet 2007. Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

M282. *Proposé par J. Walter Lynch, Athens, GA, USA.*

Quatre rectangles sont arrangés en un motif carré, de sorte qu'ils entourent un carré plus petit. Soit S l'aire du carré extérieur et Q celle du carré intérieur. Si $S/Q = 9 + 4\sqrt{5}$, déterminer le rapport des côtés des rectangles.



M283. *Proposé par Neven Jurič, Zagreb, Croatie.*

Trouver la relation entre x et y , si

$$x^2 + y \cos^2 \alpha = x \sin \alpha \cos \alpha \quad \text{et} \quad x \cos 2\alpha + y \sin 2\alpha = 0.$$

(On suppose que x et y sont tous deux non nuls.)

M284. *Proposé par Bruce Sawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Montrer que

$$\tan^{-1} \left(\frac{1}{2} \right) + \tan^{-1} \left(\frac{1}{4} \right) + \tan^{-1} \left(\frac{1}{13} \right) = \frac{\pi}{4}.$$

M285. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit a , b et c trois nombres strictement positifs tels que $a+b+c \geq 3abc$.
Montrer que $a^2 + b^2 + c^2 \geq 2abc$.

M286. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Si $xy + yz + zx = 1$, montrer que

$$(a) \quad \frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} = \frac{2}{\sqrt{(1+x^2)(1+y^2)(1+z^2)}};$$

$$(b) \quad \frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} = \frac{2}{x+y+z-xyz}.$$

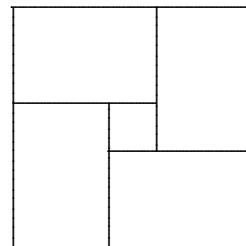
M287. *Proposé par Bruce Sawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Avec la règle et le compas, construire la moyenne harmonique de deux nombres réels donnés a et b .

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M282. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Four rectangles are arranged in a square pattern so that they enclose a smaller square. Let S be the area of the outer square and Q the area of the inner square. If $S/Q = 9 + 4\sqrt{5}$, determine the ratio of the sides of the rectangles.



M283. *Proposed by Neven Jurič, Zagreb, Croatia.*

Determine the relationship between x and y if

$$x^2 + y \cos^2 \alpha = x \sin \alpha \cos \alpha \quad \text{and} \quad x \cos 2\alpha + y \sin 2\alpha = 0.$$

(Assume that both x and y are non-zero.)

M284. Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.

Prove that

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{13}\right) = \frac{\pi}{4}.$$

M285. Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

Let a , b , and c be strictly positive numbers such that $a + b + c \geq 3abc$. Prove that $a^2 + b^2 + c^2 \geq 2abc$.

M286. Proposed by K. R. S. Sastry, Bangalore, India.

If $xy + yz + zx = 1$, show that

$$(a) \frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} = \frac{2}{\sqrt{(1+x^2)(1+y^2)(1+z^2)}};$$

$$(b) \frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} = \frac{2}{x+y+z-xyz}.$$

M287. Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.

Given two positive real numbers a and b , construct their harmonic mean with straightedge and compass.

Mayhem Solutions

M232. Proposé par Nicholas Buck, College of New Caledonia, Prince George, CB, et John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB.

On peut recouvrir un échiquier standard de 8 lignes par 8 colonnes avec 32 dominos, chaque domino couvrant deux cases adjacentes. Supposons qu'on enlève au hasard deux cases. Si le nouvel échiquier obtenu ne peut plus être recouvert par 31 dominos, quelle est la probabilité pour que :

1. les deux cases enlevées soient dans la même ligne ?
2. les deux cases enlevées se touchent en un sommet (diagonalement, horizontalement ou verticalement) ?

Solution par Jean-David Houle, Cégep de Drummondville, Drummondville, QC.

Supposons, sans perte de généralité, que les cases de l'échiquier sont alternativement noires et blanches, comme celles d'un échiquier standard. On montre tout d'abord que le nouvel échiquier ne peut plus être recouvert par 31 dominos si et seulement si les deux cases retirées sont de la même couleur.

En effet, un domino recouvre deux cases adjacentes verticalement ou horizontalement, donc les deux cases sont toujours de couleurs différentes, soit une noire et une blanche. Si on recouvre l'échiquier par 31 dominos, il doit donc y avoir 31 cases blanches et 31 cases noires. Puisque l'échiquier original contient 32 cases de chaque couleur, les deux cases retirées doivent être de couleurs différentes. [*Réd* : De plus, si les deux cases retirées sont de couleurs différentes, elles sont aux coins opposés d'un rectangle de dimensions pair par impair et ce rectangle peut toujours être recouvert par des dominos. En séparant l'échiquier avec des tranches dans la direction de la dimension impaire du rectangle, on obtient des rectangles qui ont tous au moins une dimension paire et qui peuvent donc tous être recouverts avec des dominos.] Ceci indique que si on ne peut plus recouvrir l'échiquier par 31 dominos, les deux cases retirées sont de la même couleur.

1. Une ligne contient 4 cases de la même couleur, alors après avoir retiré la première case, il reste 3 cases de la même couleur sur la même ligne. Donc la probabilité est

$$\frac{\text{cas favorables}}{\text{cas possibles}} = \frac{3}{31} \approx 0,1935.$$

2. Deux cases qui se touchent en un sommet verticalement ou horizontalement sont nécessairement de couleurs différentes, donc les deux cases retirées se touchent diagonalement. On note que pour une couleur donnée, 18 cases ont 4 contacts diagonaux, 12 cases ont 2 contacts diagonaux et 2 cases ont un contact diagonal. Donc la probabilité est

$$\frac{\text{cas favorables}}{\text{cas possibles}} = \frac{18 \cdot 4 + 12 \cdot 2 + 2 \cdot 1}{32 \cdot 31} = \frac{98}{992} \approx 0,0988.$$

Autre solution soumise par Richard I. Hess, Rancho Palos Verdes, CA, É-U.

M233. *Proposed by Richard K. Guy, University of Calgary, Calgary, AB.*

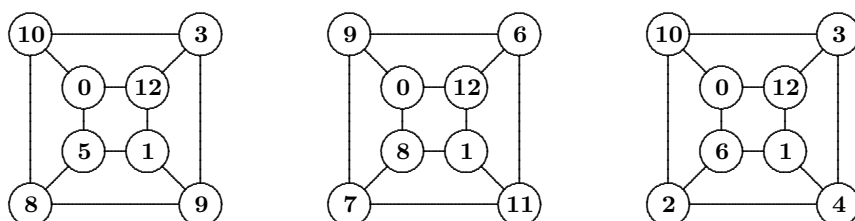
Can you place eight distinct integers selected from 0 to 12 at the vertices of a cube so that the twelve edges have the differences 1, 2, ..., 12 between their end-points?

Either find a way to do this, or prove that it is impossible.

Solution by the proposer, modified by the editor.

Since 12 is one of the differences, the numbers 0 and 12 must be on two adjacent vertices of the cube. Since 11 is another difference, either a

vertex labelled 11 is adjacent to the vertex labelled 0 or a vertex labelled 1 is adjacent to the vertex labelled 12. However, these two possibilities are related by the mapping $f(n) = 12 - n$. Let us assume that a vertex labelled 1 is adjacent to the vertex labelled 12. Here are three such distinct cubes :



Also solved by Richard I. Hess, Rancho Palos Verdes, CA, USA. The proposer suspects that there may be as many as a dozen or more distinct solutions.

M234. *Proposé par K. R. S. Sastry, Bangalore, Inde.*

Soit J un nombre de deux chiffres sans communs diviseurs autres que 1. En permutant ces deux chiffres, on obtient un nombre I qui est $p\%$ plus grand que J . Trouver toutes les valeurs possibles de p , p étant un nombre naturel positif plus petit que 100.

Solution par Jean-David Houle, Cégep de Drummondville, Drummondville, QC, modifié par le rédacteur.

Soit $J = 10x + y$ où x et y sont des entiers entre 1 et 9, avec $(x, y) = 1$. Donc $I = 10y + x$. Aussi, $I = \left(1 + \frac{p}{100}\right) J$. En isolant p , on obtient :

$$p = 100 \cdot \frac{I - J}{J} = \frac{900(y - x)}{J}. \quad (1)$$

Puisque p est un nombre naturel, alors $y > x$ et $10x + y \mid 900(y - x)$.

Supposons que J et $y - x$ ont un diviseur commun plus grand que 1. Alors ils ont un diviseur principal commun q . Puisque $q \mid J = 10x + y$ et $q \mid y - x$, on a

$$q \mid (10x + y) - (y - x) = 11x.$$

Puisque $q < y - x \leq 9$, on obtient $q \mid x$. Donc $q \mid (y - x) + x = y$, qui est une contradiction, parce qu'ils n'ont aucun diviseur commun plus grand que 1. Ainsi, $(J, y - x) = 1$.

Donc $J \mid 900$. Les valeurs possibles pour J sont $\{12, 15, 18, 25, 45\}$. En remplaçant ces valeurs dans (1), on trouve $p \in \{75, 240, 350, 108, 20\}$. Puisque $p < 100$, on a $p = 20$ et $p = 75$.

En outre résolu par Michelle Ellenburg et Christopher Odom, étudiants, Angelo State University, San Angelo, TX, É-U; et Richard I. Hess, Rancho Palos Verdes, CA, É-U.

M235. *Proposé par Ron Lancaster, Université de Toronto, Toronto, ON.*

Résoudre l'équation

$$2^x + 2^{x+1} + \dots + 2^{x+2006} = 4^x + 4^{x+1} + \dots + 4^{x+2006}.$$

Solution par Jean-David Houle, Cégep de Drummondville, Drummondville, QC.

On simplifie l'équation :

$$\begin{aligned} 2^x(1 + 2^1 + \dots + 2^{2006}) &= 4^x(1 + 4^1 + \dots + 4^{2006}), \\ 1 + 2^1 + \dots + 2^{2006} &= 2^x(1 + 4^1 + \dots + 4^{2006}). \end{aligned}$$

On remplace les séries géométriques par leur sommes :

$$\begin{aligned} 1 + 2^1 + \dots + 2^{2006} &= \frac{1 - 2^{2007}}{1 - 2} = 2^{2007} - 1 \\ \text{et } 1 + 4^1 + \dots + 4^{2006} &= \frac{1 - 4^{2007}}{1 - 4} = \frac{4^{2007} - 1}{3}. \end{aligned}$$

Donc :

$$\begin{aligned} 1 + 2^1 + \dots + 2^{2006} &= 2^x(1 + 4^1 + \dots + 4^{2006}), \\ 2^{2007} - 1 &= 2^x \cdot \frac{4^{2007} - 1}{3}, \\ 2^x &= \frac{(3)(2^{2007} - 1)}{4^{2007} - 1} = \frac{(3)(2^{2007} - 1)}{(2^{2007} + 1)(2^{2007} - 1)} = \frac{3}{2^{2007} + 1}. \end{aligned}$$

On peut en approximer le résultat :

$$x = \log_2\left(\frac{3}{2^{2007} + 1}\right) \approx \log_2\left(\frac{3}{2^{2007}}\right) = \log_2 3 - 2007 \approx -2005,415.$$

En outre résolu par RICHARD I. HESS, Rancho Palos Verdes, CA, É-U; et JOSH TREJO et MANDY RODGERS, étudiants, Angelo State University, San Angelo, TX, É-U.

M236. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

A traveller to a strange island discovers that it is inhabited by knights who can only make true statements and knaves who can only make false statements. One day a traveller encountered three inhabitants, whom we will call *A*, *B*, and *C*, and asked, "How many knights are there among you three?"

A made an answer, which the traveller missed, but which was understood by the other two. When *B* was asked what *A* said, *B* responded, "A said that there is one knight among us."

"Don't believe *B*," exclaimed *C*, "he is lying."

What are *B* and *C*?

Solution by Mandy Rodgers and Josh Trejo, students, Angelo State University, San Angelo, TX, USA.

Since B and C made contradictory statements, they cannot both be knaves, or both knights. Thus, there are two cases to consider: B is a knight and C is a knave, or B is a knave and C is a knight.

Assume first that B is a knight and is telling the truth (and C is a knave). Then A really did say that there is one knight. Now, if A were a knight, his statement "There is one knight among us" would need to be true and would lead to a contradiction, since both A and B would be knights. If A were a knave, his statement "There is one knight among us" would need to be false, which would again lead to a contradiction, since B would be the one and only knight. Hence, B cannot be a knight.

Therefore, B is a knave and is telling a lie (and C is a knight), which answers the question asked. We should explore this possibility to see if it can actually occur. Assume that B is a knave and C is a knight. Then A did not say that there is one knight. Now, if A were a knight, he could have said that there were 2 knights, which would be consistent, since both A and C would be knights. If A were a knave, he could have said they were all knights or all knaves. Thus, although it is not possible to determine whether A is a knight or a knave, we do know that B is a knave and C is a knight.

Also solved by JEAN-DAVID HOULE, Cégep de Drummondville, Drummondville, QC; and MICHELLE ELLENBURG and CHRISTOPHER ODOM, students, Angelo State University, San Angelo, TX, USA.

M237. *Proposed by K.R.S. Sastry, Bangalore, India.*

Let ABC be an isosceles triangle with $AB = AC$, and let the lengths of the sides be integers with no common divisor other than 1. The incentre I divides the internal angle bisector AD such that $\frac{AI}{ID} = \frac{25}{24}$. Determine the radius of the incircle of $\triangle ABC$.

Solved by Richard I. Hess, Rancho Palos Verdes, CA, USA.

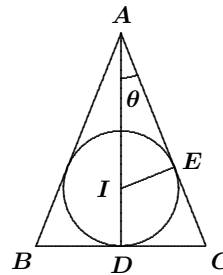
Let a and b be relatively prime integers such that $a = BC$ and $b = AC = AB$, and let $\theta = \angle DAC$. We know that $\sin \theta = \frac{1}{2}a/b$ and $\sin \theta = IE/AI$. Since $IE = ID$, we conclude that

$$\sin \theta = \frac{a}{2b} = \frac{24}{25}.$$

Hence, since a and b are relatively prime, we have $a = 48$ and $b = 25$. Then $EC = DC = \frac{1}{2}a = 24$; thus, $AE = AC - EC = 1$. Now,

$$\begin{aligned} r &= IE = AE \tan \theta = \tan \theta = \frac{\sin \theta}{\cos \theta} \\ &= \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}} = \frac{24/25}{7/25} = \frac{24}{7}. \end{aligned}$$

There was one incorrect solution submitted.



Problem of the Month

Ian VanderBurgh

This month, we have a couple of problems demonstrating that knowing too much algebra may be dangerous!

Problem #1 (1974 Gauss Contest). A car is driven up a 1 km long hill at 30 km/h, and continues down the other side, which is also 1 km in length. The speed the car must be driven on the down slope, in km/h, in order to average 60 km/h for the whole trip is

- (A) 30 (B) 90 (C) 60 (D) 120 (E) none of these

It is tempting to answer 90 km/h, since the average of 30 and 90 is 60, but this somehow seems too easy. (To boot, you probably have that nagging voice in the back of your head reminding you of something your Grade 5 mathematics teacher told you about this sort of problem. . . .)

The most important thing to remember in solving this problem is that speed equals distance divided by time, which is the same as distance equals speed multiplied by time, or time equals distance divided by speed.

Solution #1: To drive up the 1 km hill at 30 km/h takes $\frac{1}{30}$ hour, or 2 minutes. To average 60 km/h over the whole 2-km trip, the total driving time must be $\frac{2}{60}$ hour, or 2 minutes. Then the downhill part of the trip must take $2 - 2 = 0$ minutes. Since this is not possible, the answer must be (E).

Now, that was a surprising answer!

If we know some algebra (and try to use it), the solution becomes a bit more complicated.

Solution #2: Suppose the car is driven down the hill at v km/h. To find the average speed, we find the total distance driven (2 km in this case) and divide by the total time. The time for the uphill portion is $\frac{1}{30}$ hour as in Solution #1 above. The time for the downhill portion is $1/v$ hour. Therefore, the total time is $\frac{1}{30} + \frac{1}{v}$ hour. Hence, the average speed, in km/h, is

$$\frac{2}{\frac{1}{30} + \frac{1}{v}} = 60,$$

which simplifies to

$$2 = 60 \left(\frac{1}{30} + \frac{1}{v} \right) = 2 + \frac{60}{v}.$$

Thus, $60/v = 0$, which is impossible.

Let's try a couple of variations of the problem. First, try the problem with 40 km/h instead of 30 km/h. Try it by both of the methods used above. Did you get 120 km/h? (Going uphill for 1 km at 40 km/h should take $\frac{1}{40}$ of an hour, which is $1\frac{1}{2}$ minutes, leaving half a minute out of the total of 2 minutes to drive the remaining 1 km downhill.)

What happens if we replace the 30 km/h with 20 km/h? If we proceed mathematically (without thinking), we find that the car must be driven down the hill at -60 km/h. One wonders what that really means! This shows that we always need to think about what we are doing.

Suppose the car is driven uphill at u km/h and then downhill at v km/h. What are the possible values of u that allow us to solve the problem (that is, to get a positive value for v)?

We model Solution #2 from above. The time to drive uphill is $1/u$ hours and to drive downhill is $1/v$ hours. Therefore, the total driving time is $(1/u) + (1/v)$ hours, and the average speed is

$$\frac{2}{\frac{1}{u} + \frac{1}{v}} = 60.$$

Now we solve for v in terms of u :

$$\begin{aligned} 2 &= 60 \left(\frac{1}{u} + \frac{1}{v} \right), \\ \frac{1}{30} - \frac{1}{u} &= \frac{1}{v}, \\ v &= \frac{1}{\frac{1}{30} - \frac{1}{u}} = \frac{30u}{u - 30}. \end{aligned}$$

We know that u is a positive real number. For v to be a positive real number, we need $u - 30 > 0$, or $u > 30$. Thus, an uphill speed of more than 30 km/h allows us to find a downhill speed that gives an average speed of 60 km/h.

For what *integer* values of u can we find a positive integer value of v that gives an average speed of 60 km/h? We know already that

$$v = \frac{30u}{u - 30} = \frac{30u - 900 + 900}{u - 30} = 30 + \frac{900}{u - 30}.$$

For v to be an integer, we need $\frac{900}{u - 30}$ to be an integer, and for this we need $u - 30$ to be a divisor of 900. You can list out the divisors of 900 and the corresponding values of u .

So, a problem that starts out being quite simple has lots of interesting ideas that can be gleaned from it. The most important thing to remember here is the very first simple solution. Thinking about this type of problem in a clever way will often get you the answer more easily than using a formal algebraic approach.

Let's try to apply this way of thinking to another problem.

Problem #2. Jeff is on a railway bridge joining A to B , and is $\frac{3}{8}$ of the way across from A . He hears a train approaching A ; it is travelling at 80 km/h. If he runs towards A , he will meet the train at A . If he runs towards B , the train will overtake him at B . How fast can he run?

Solution #1: Let's try this algebraically first. Suppose the bridge has a length R , the train is a distance d from point A , and Jeff's running speed is v . The amount of time it would take Jeff to run to A is $\frac{3}{8}R/v$ and to run to B is $\frac{5}{8}R/v$. (Each of these quantities is distance divided by speed.) The amount of time it would take the train to get to A is $d/80$ and to get to B is $(d + R)/80$. Since Jeff and the train would arrive at A or B at the same time,

$$\frac{\frac{3}{8}R}{v} = \frac{d}{80} \quad \text{and} \quad \frac{\frac{5}{8}R}{v} = \frac{d + R}{80}.$$

Now we have two equations with three unknowns and want to solve for v . Try fiddling with these before going on!

—Any luck? We could solve this by substituting one equation into the other, but here's a more clever way. By subtracting the first equation from the second one, we get

$$\frac{\frac{1}{4}R}{v} = \frac{R}{80},$$

which yields $v = 20$ km/h. Hence, Jeff runs at 20 km/h.

But perhaps we can find a better method. How about this?

Solution #2: In the amount of time that Jeff runs $\frac{3}{8}$ of the way across the bridge, the train gets to A . Suppose that Jeff runs $\frac{3}{8}$ of the way across the bridge towards B , not A . Once Jeff has run this distance, he is $\frac{3}{8} + \frac{3}{8} = \frac{3}{4}$ of the way from A to B , and the train is at A . But we also know that Jeff and the train get to B at the same time with Jeff running in this direction. Therefore, the train will travel the length of the entire bridge while Jeff runs the remaining $\frac{1}{4}$ of the length of the bridge. Thus, the train's speed is 4 times Jeff's speed; that is, Jeff's speed is $\frac{80}{4} = 20$ km/h.

Isn't that nice? To finish off, here is a problem of the same type for you to try:

A train passes completely through a tunnel in 10 minutes. A second train, twice as long, passes through the tunnel in 11 minutes. If both trains are travelling at the same speed, 72 km/h, determine the length of the tunnel and the lengths of the trains.

THE OLYMPIAD CORNER

No. 260

R.E. Woodrow

We begin this number with the problems of the Second Round and Final Round of the Hungarian National Olympiad Grades 11–12 for 2003–2004. Thanks again go to Christopher Small, Canadian Team leader to the IMO in Athens, Greece in 2004, for collecting them for our use.

HUNGARIAN NATIONAL OLYMPIAD 2003–2004 Grades 11–12, Round 2

1. Let n be an integer, $n > 1$. Define

$$A = \frac{\sqrt{n+1}}{n} + \frac{\sqrt{n+4}}{n+3} + \frac{\sqrt{n+7}}{n+6} + \frac{\sqrt{n+10}}{n+9} + \frac{\sqrt{n+13}}{n+12}$$

and

$$B = \frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{n+2}} + \frac{1}{\sqrt{n+5}} + \frac{1}{\sqrt{n+8}} + \frac{1}{\sqrt{n+11}}.$$

Determine which of the following relations holds (depending on n): $A > B$, $A = B$, or $A < B$.

2. Let a , b , and c denote the sides of a triangle opposite the angles A , B , and C , respectively. Let r be the inradius and R the circumradius of the triangle. If $\angle A \geq 90^\circ$, prove that

$$\frac{r}{R} \leq \frac{a \sin A}{a + b + c}.$$

3. Prove that the equation $x^3 + 2px^2 + 2p^2x + p = 0$ cannot have three distinct real roots, for any real number p .

4. Let $ABCD$ be a cyclic quadrilateral with $AB = 2AD$ and $BC = 2CD$. Let $d = AC$ and $\alpha = \angle BAD$ be given. Express the area of $ABCD$ in terms of d and α .

Grades 11–12, Final Round

1. Let ABC be an acute triangle, and let P be a point on side AB . Draw lines through P parallel to AC and BC , and let them cut BC and AC at X and Y , respectively. Construct (with straightedge and compass) the point P which gives the shortest length XY . Prove that the shortest XY is perpendicular to the median of ABC through C .

2. Let a , b , and c be distinct positive integers which are the side lengths of a triangle. There is a line which cuts both the area and the perimeter of the triangle into two equal parts. This line cuts the longest side of the triangle into two parts with ratio 2 : 1. Determine a , b , and c for which abc is minimal.

3. Let $H = \{1, 2, 3, \dots, 2004\}$. We denote by D the number of subsets of H such that the sum of the elements of the subset has a remainder of 7 when divided by 32. We denote by S the number of subsets of H such that the sum of the elements of the subset has a remainder of 14 when divided by 16. Prove that $S = 2D$.

Next we give the problems of the First Round (Specialized Mathematics Classes), Grades 11–12 of the Hungarian National Olympiad. Thanks again go to Christopher Small, Canadian Team Leader to the IMO in Athens, Greece in 2004, for collecting the set.

HUNGARIAN NATIONAL OLYMPIAD 2003–2004 (Specialized Mathematics Classes) Grades 11–12

First Round

1. Let n be a positive integer, and let a and b be positive real numbers. Prove that

$$\begin{aligned} \log(a^n) + \binom{n}{1} \log(a^{n-1}b) + \binom{n}{2} \log(a^{n-2}b^2) + \dots + \log(b^n) \\ = \log((ab)^{n2^{n-1}}). \end{aligned}$$

2. Let H be a finite set of positive integers none of which has a prime factor greater than 3. Show that the sum of the reciprocals of the elements of H is smaller than 3.

3. Consider the three disjoint arcs of a circle determined by three points on the circle. For each of these arcs, draw a circle at the mid-point of the arc and passing through the end-points of the arc. Prove that the three circles have a common point.

4. A palace which has a square shape is divided into 2003×2003 square rooms of the same size which form a square grid. There might be a door between two rooms if they have a common side. The main gate leads to the room at the northwest corner. Someone has entered the palace, walked around for a while and upon returning to the room at the northwest corner for the first time, immediately left the palace. It turned out that this person visited each of the other rooms 100 times, except the room at the southeast corner. How many times did this person visit the room at the southeast corner?

5. Let $a_0, a_1, \dots, a_n, a_{n+1}$ be real numbers such that $a_0 = a_{n+1} = 0$. Prove that there is a number k ($0 \leq k \leq n$) such that

(a) $a_{k+1} + \dots + a_{k+i} \geq 0$ for every $i = 1, \dots, n - k + 1$, and

(b) $a_j + \dots + a_k \leq 0$ for every $j = 0, \dots, k$.

To complete the problems section we give the Final Round of the Finnish High School Mathematics Contest. My thanks go to Matti Lehtinen, Helsinki, Finland; and to Christopher Small, Canadian Team Leader to the IMO in Athens, Greece in 2004, for collecting them for our use.

FINNISH HIGH SCHOOL MATH CONTEST 2004 Final Round

February 6, 2004 – Time allowed: 3 hours

1. The equations

$$x^2 + 2ax + b^2 = 0 \quad \text{and} \quad x^2 + 2bx + c^2 = 0$$

both have two different real roots. Determine the number of real roots of the equation

$$x^2 + 2cx + a^2 = 0.$$

2. Let a, b , and c be positive integers such that

$$\frac{a\sqrt{3} + b}{b\sqrt{3} + c}$$

is a rational number. Show that

$$\frac{a^2 + b^2 + c^2}{a + b + c}$$

is an integer.

3. Two circles with radii r and R are externally tangent at a point P . Determine the length of the segment cut from the common tangent through P by the other common tangents.

4. The numbers $2005! + 2, 2005! + 3, \dots, 2005! + 2005$ form a sequence of 2004 consecutive integers, none of which is a prime number. Does there exist a sequence of 2004 consecutive integers containing exactly 12 prime numbers?

5. Finland is going to change its monetary system again and replace the Euro by the Finnish Mark. The Mark is divided into 100 pennies. There shall be coins of three denominations only, and the number of coins a person has to carry in order to be able to pay for any purchase less than one Mark should be minimal. Determine the coin denominations.

Now we turn to readers' solutions to problems given in the December 2005 number of the *Corner*, for the 38th Mongolian Mathematical Olympiad, Final Round, appearing [2006 : 505].

1. Let n and k be natural numbers. Find the least possible value for the cardinality of a set A that satisfies the following condition: There exist subsets A_1, \dots, A_n of A such that any union of k of the A_i is equal to A , but any union of $k - 1$ of them is not equal to A .

Comment by Pierre Bornsztein, Maisons-Laffitte, France.

The answer is $\binom{n}{k-1}$.

This problem is equivalent to problem #6 of the final round of the 8th Korean Mathematical Olympiad. A proof appears in [2000 : 11].

2. For a natural number p , one can move between two integer points in a plane when the distance between the points is p . Find all primes p for which the point $(2002, 38)$ can be reached from the point $(0, 0)$ using permitted moves.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

The desired primes are those not of the form $4k + 3$.

Let p be a prime.

Case 1. $p \equiv 3 \pmod{4}$.

It is well known that p is not the sum of two squares. Therefore, moves can only be done in the directions of the coordinate axes. In that case, both coordinates of any point which can be reached must be multiples of p . But $2002 = 2 \times 7 \times 11 \times 13$ and $38 = 2 \times 19$ have no common prime divisor of the form $4k + 3$. Thus, it is impossible to reach $(2002, 38)$ for such a p .

Case 2. $p = 2$.

One can reach $(2002, 38)$ by 1001 moves of the form $(x, y) \rightarrow (x+2, y)$, and 19 moves of the form $(x, y) \rightarrow (x, y+2)$.

Case 3. $p \equiv 1 \pmod{4}$.

It is well known that there exist two positive integers a and b such that $p = a^2 + b^2$. Moreover, since p is prime, we must have a and b coprime, and one of them, say a , is even.

Let us consider moves of the following four types:

1. $(x, y) \rightarrow (x + a, y + b)$,
2. $(x, y) \rightarrow (x + a, y - b)$,
3. $(x, y) \rightarrow (x + b, y + a)$,
4. $(x, y) \rightarrow (x + b, y - a)$.

We will prove that we can reach $(2002, 38)$ from $(0, 0)$ using only these four moves and their inverses (where the inverse of $(x, y) \rightarrow (x + a, y + b)$ is $(x, y) \rightarrow (x - a, y - b)$).

Let x_i denote the number of moves of type i (where x_i may be negative, meaning that we use the inverse of the move of type i $|x_i|$ times). We want to prove that there exist integers x_1, x_2, x_3, x_4 such that

$$2002 = a(x_1 + x_2) + b(x_3 + x_4) \quad \text{and} \quad 38 = a(x_3 - x_4) + b(x_1 - x_2).$$

But, since a and b are coprime, there exist integers $\alpha, \beta, \gamma, \delta$ such that $2002 = a\alpha + b\beta$ and $38 = a\gamma + b\delta$. Note that β and δ are even. Then, for all integers m and n , we have

$$2002 = a(\alpha + bm) + b(\beta - am) \quad \text{and} \quad 38 = a(\gamma + nb) + b(\delta - na).$$

Hence, we want to find integer solutions to the system

$$\begin{aligned} x_1 + x_2 &= \alpha + bm, \\ x_1 - x_2 &= \delta - na, \\ x_3 + x_4 &= \beta - am, \\ x_3 - x_4 &= \gamma + nb. \end{aligned}$$

But the only condition which has to be satisfied to solve this system in the integers is that $\alpha + bm$ and $\gamma + nb$ are even. Since b is odd, this can be done by suitable choices for m and n , and we are done.

4. Given are 131 distinct natural numbers, each with prime divisors not exceeding 42. Prove that four of them can be chosen whose product is a perfect square.

Solution by Pierre Bornsstein, Maisons-Laffitte, France.

Let x_1, \dots, x_{131} be the given numbers. Direct checking shows that there are exactly 13 primes not exceeding 42. Denote them by p_1, \dots, p_{13} .

Consider the $\frac{131 \times 130}{2} = 8515$ products $x_i x_j$ (with repetition if any) of any two of the given numbers. Consider each of these products, say P , as a 13-tuple (a_1, \dots, a_{13}) , where a_i is the exponent, reduced modulo 2, of p_i in the prime decomposition of P (thus, $a_i \in \{0, 1\}$). Since $8515 > 8192 = 2^{13}$, two of these products, say $x_i x_j$ and $x_m x_n$, must be associated with the same 13-tuple. If $\{i, j\} \cap \{m, n\} = \emptyset$, this ensures that $x_i x_j x_m x_n$ is a perfect square, and we are done.

Otherwise, without loss of generality, we may assume that $j = n$, which means that $x_i x_m$ is associated with $(0, 0, \dots, 0)$. In that case, omit x_i and x_m from the given numbers and repeat the above procedure. Since $\frac{129 \times 128}{2} = 8256 > 2^{13}$, we may find again two products, say $x_r x_s$ and $x_t x_u$ which are associated with the same 13-tuple. If $\{r, s\} \cap \{t, u\} = \emptyset$, this ensures that $x_r x_s x_t x_u$ is a perfect square, and we are done. Otherwise, without loss of generality, we may assume that $s = u$, which means that $x_r x_t$ is associated with $(0, 0, \dots, 0)$. Then $x_i x_m x_r x_t$ is the desired square.

5. Let a_0, a_1, \dots be an infinite sequence of positive real numbers. Show that $1 + a_n > \sqrt[n]{2} a_{n-1}$ for infinitely many positive integers n .

Solution by Pierre Bornsstein, Maisons-Laffitte, France.

Suppose, for the purpose of contradiction, that there exists $n_0 \geq 0$ such that $1 + a_n \leq \sqrt[n]{2} a_{n-1}$ for all $n > n_0$. We have $\sqrt[n]{2} \leq 1 + \frac{1}{n}$, from Bernoulli's Inequality. Hence, for all $n > n_0$,

$$a_n \leq \frac{n+1}{n} a_{n-1} - 1. \quad (1)$$

We prove by induction on $p \geq 1$ that

$$a_{n_0+p} \leq (n_0 + p + 1) \left(\frac{a_{n_0}}{n_0 + 1} - \sum_{k=n_0+2}^{n_0+p+1} \frac{1}{k} \right). \quad (2)$$

This is true for $p = 1$, because in this case it is just (1) with $n = n_0 + 1$. Now let us assume it holds for some given $p \geq 1$. Using (1) with $n = n_0 + p + 1$ and then applying the induction hypothesis, we get

$$\begin{aligned} a_{n_0+p+1} &\leq \frac{n_0 + p + 2}{n_0 + p + 1} a_{n_0+p} - 1 \\ &\leq (n_0 + p + 2) \left(\frac{a_{n_0}}{n_0 + 1} - \sum_{k=n_0+2}^{n_0+p+1} \frac{1}{k} \right) - 1 \\ &= (n_0 + p + 2) \left(\frac{a_{n_0}}{n_0 + 1} - \sum_{k=n_0+2}^{n_0+p+2} \frac{1}{k} \right), \end{aligned}$$

which ends the induction.

It is well known that $\sum \frac{1}{k}$ diverges to $+\infty$. For sufficiently large p ,

$$\sum_{k=n_0+2}^{n_0+p+1} \frac{1}{k} > \frac{a_{n_0}}{n_0 + 1}.$$

For such a p , the inequality (2) forces a_{n_0+p} to be negative, a contradiction.

Next we look at a solution from a reader to problem 1 of the 19th Balkan Mathematical Olympiad, which appeared [2005 : 506].

1. Let A_1, A_2, \dots, A_n ($n \geq 4$) be points in the plane such that no three of them are collinear. Some pairs of distinct points among A_1, A_2, \dots, A_n are connected by line segments in such a way that each point is connected to at least three others. Prove that there exists $k > 1$ and distinct points $X_1, X_2, \dots, X_{2k} \in \{A_1, A_2, \dots, A_n\}$ such that for each $1 \leq i \leq 2k - 1$, X_i is connected to X_{i+1} and X_{2k} is connected to X_1 .

Solution by Pierre Bornsstein, Maisons-Laffitte, France.

An equivalent formulation is the following: A finite simple graph for which each vertex has degree at least 3 contains an even cycle.

Consider the longest path (using pairwise distinct vertices) in the graph, say X_1, X_2, \dots, X_p . According to the maximality of the path, each vertex adjacent to X_1 must belong in the path. Since X_1 has degree at least 3, X_1 is adjacent to X_r and X_s , where $2 < r < s$.

Now, among the three integers 2, r , and s , at least two have the same parity. Then, X_1 is adjacent to X_a and to X_b , where $2 \leq a < b \leq p$ and $a \equiv b \pmod{2}$. Thus, $X_1-X_a-X_{a+1}-\dots-X_b-X_1$ is the desired even cycle.

Now we move to solutions from readers to problems of the Bulgarian Mathematical Olympiad, Final Round, 2003, given [2005 : 506–507].

3. Given the sequence $\{y_n\}_{n=1}^{\infty}$ defined by $y_1 = y_2 = 1$ and

$$y_{n+2} = (4k - 5)y_{n+1} - y_n + 4 - 2k, \quad n \geq 1,$$

find all integers k such that every term of the sequence is a perfect square.

Solved by Robert Bilinski, Collège Montmorency, Laval, QC; and Pierre Bornsztejn, Maisons-Laffitte, France. We give the write-up of Bornsztejn.

The desired values are $k = 1$ and $k = 3$.

Assume that k is an integer such that $\{y_n\}_{n=1}^{\infty}$ contains only perfect squares. Then $y_3 = 2k - 2$ is an even square, say $4a^2$. Thus, $k = 2a^2 + 1$ for some non-negative integer a .

Moreover, the recurrence relation yields $y_4 = 8k^2 - 20k + 13$ and $y_5 = 32k^3 - 120k^2 + 148k - 59$. Thus, $y_5 = 256a^6 - 96a^4 + 8a^2 + 1$.

But, if $a \geq 2$, we have

$$256a^6 - 96a^4 + 8a^2 + 1 < 256a^6 - 96a^4 + 9a^2 = (16a^2 - 3a)^2,$$

while, since $a(32a^2 - a - 6) > 0$,

$$\begin{aligned} 256a^6 - 96a^4 + 8a^2 + 1 &> 256a^6 - 96a^4 - 32a^3 + 9a^2 + 6a + 1 \\ &= (16a^2 - 3a - 1)^2. \end{aligned}$$

Thus, $(16a^3 - 3a - 1)^2 < y_5 < (16a^3 - 3a)^2$, which contradicts the fact that y_5 is square.

Therefore, $a \in \{0, 1\}$, which leads to $k = 1$ or $k = 3$.

Conversely, consider the cases $k = 1$ and $k = 3$.

Case 1. $k = 1$.

One can verify by induction that the sequence $\{y_n\}_{n=1}^{\infty}$ is 1, 1, 0, 1, 1, 0, 1, 1, 0, ... which is periodic with period 3. Since it contains only squares, $k = 1$ is a solution of the problem.

Case 2. $k = 3$.

Then $y_{n+2} = 7y_{n+1} - y_n - 2$ for $n \geq 1$. We will prove that, for all $n \geq 1$, we have $y_n = x_n^2$ where

$$x_1 = x_2 = 1 \quad \text{and} \quad x_{n+2} = 3x_{n+1} - x_n. \quad (1)$$

First, we prove that for the sequence $\{x_n\}_{n=1}^{\infty}$ defined by (1), we have

$$x_{n+1}^2 + x_n^2 + 1 = 3x_{n+1}x_n \quad \text{for all } n \geq 1. \quad (2)$$

This is clearly true for $n = 1$. Assume that it holds for some given $n \geq 1$. Then

$$\begin{aligned} x_{n+2}^2 + x_{n+1}^2 + 1 &= x_{n+2}^2 + 3x_{n+1}x_n - x_n^2 \quad (\text{induction hypothesis}) \\ &= x_{n+2}^2 + x_n(3x_{n+1} - x_n) \\ &= x_{n+2}^2 + x_nx_{n+2} \quad \text{from (1)} \\ &= x_{n+2}(x_{n+2} + x_n) = 3x_{n+2}x_{n+1} \quad \text{from (1)}. \end{aligned}$$

This proves the relation for $n + 1$, and ends the induction.

It follows that, for all $n \geq 1$, we have:

$$\begin{aligned} x_{n+2}^2 &= (3x_{n+1} - x_n)^2 = 7x_{n+1}^2 + (2x_{n+1}^2 + x_n^2 - 6x_{n+1}x_n) \\ &= 7x_{n+1}^2 - x_n^2 - 2 \quad \text{from (2)}. \end{aligned}$$

Hence, $\{x_n^2\}_{n=1}^{\infty}$ satisfies the same recurrence relation as does $\{y_n\}_{n=1}^{\infty}$. Since $y_1 = x_1^2$ and $y_2 = x_2^2$, it easily follows by induction that $y_n = x_n^2$ for all $n \geq 1$. Thus, $k = 3$ is a solution of the problem, and we are done.

Remark. Using (2), we can prove that $x_n = f_{2n+1}$ for all $n \geq 1$, where $\{f_n\}$ is the Fibonacci sequence.

Next we move to the February 2006 number of the *Corner* and solutions from readers to problems of the 2003 Vietnamese Mathematical Olympiad, given [2006 : 25–27].

3. Find all polynomials $P(x)$ with real coefficients, satisfying the relation

$$(x^3 + 3x^2 + 3x + 2)P(x - 1) = (x^3 - 3x^2 + 3x - 2)P(x)$$

for every real number x .

Solution by Michel Bataille, Rouen, France.

Let P be a real polynomial satisfying the given condition; that is,

$$(x + 2)(x^2 + x + 1)P(x - 1) = (x - 2)(x^2 - x + 1)P(x). \quad (1)$$

Since the polynomials $x + 2$, $x^2 + x + 1$, $x - 2$, and $x^2 - x + 1$ are irreducible over \mathbb{R} , we see that $P(x)$ is divisible by $(x + 2)(x^2 + x + 1)$. Thus,

$$P(x) = (x + 2)(x^2 + x + 1)Q(x) \quad (2)$$

for some real polynomial $Q(x)$, and (1) yields

$$P(x - 1) = (x - 2)(x^2 - x + 1)Q(x). \quad (3)$$

Taking $x = 2$ in (3) gives $P(1) = 0$, which implies that $P(x)$ is divisible by $x - 1$. Since $P(-2) = 0$ (in view of (2)), taking $x = -1$ in (3) gives $Q(-1) = 0$. Therefore, $Q(x)$ is divisible by $x + 1$, and then so is $P(x)$. Lastly, taking $x = 1$ in (1) gives $P(0) = 0$, so that $P(x)$ is divisible by x as well. Summing up, we see that $P(x) = (x+2)(x+1)x(x-1)(x^2+x+1)S(x)$ for some real polynomial $S(x)$.

Now, substituting into (1), we obtain $S(x) = S(x - 1)$ for all $x \in \mathbb{R}$, and an immediate induction shows that $S(n) = S(0)$ for all non-negative integers n . Thus, the polynomial $S(x) - S(0)$ has infinitely many roots. It follows that $S(x)$ is a constant polynomial.

Conversely, substituting $P(x) = k(x+2)(x+1)x(x-1)(x^2+x+1)$ in (1) leads to an identity, for all $k \in \mathbb{R}$.

In conclusion, the solutions of the problem are the polynomials of the form $P(x) = k(x+2)(x+1)x(x-1)(x^2+x+1)$, where $k \in \mathbb{R}$.

4. Let $P(x) = 4x^3 - 2x^2 - 15x + 9$ and $Q(x) = 12x^3 + 6x^2 - 7x + 1$.

- (i) Prove that each of these polynomials has three distinct real roots.
- (ii) Let α and β be the greatest roots of $P(x)$ and $Q(x)$, respectively. Prove that $\alpha^2 + 3\beta^2 = 4$.

Solution by Michel Bataille, Rouen, France.

(i) The polynomial P is a continuous function, and it is easily checked that $P(-2) < 0$, $P(-\frac{15}{8}) > 0$, $P(0) > 0$, $P(1) < 0$, and $P(\frac{15}{8}) > 0$. Hence, P has three distinct roots α , α_1 , and α_2 satisfying

$$\alpha \in (1, \frac{15}{8}), \quad \alpha_1 \in (0, 1), \quad \alpha_2 \in (-2, -\frac{15}{8}). \quad (1)$$

Similarly, Q has three distinct roots, β , β_1 , and β_2 such that

$$\beta \in (\frac{1}{3}, 1), \quad \beta_1 \in (0, \frac{1}{3}), \quad \beta_2 \in (-2, -1). \quad (2)$$

(ii) A polynomial $S(x)$ whose roots are exactly α^2 , α_1^2 , and α_2^2 is readily obtained by taking S such that $S(x^2) = -P(x) \cdot P(-x)$. Here, since

$$\begin{aligned} S(x^2) &= -(9 - 2x^2 + x(4x^2 - 15))(9 - 2x^2 - x(4x^2 - 15)) \\ &= x^2(4x^2 - 15)^2 - (9 - 2x^2)^2, \end{aligned}$$

we obtain $S(x) = 16x^3 - 124x^2 + 261x - 81$. Similarly, the roots of the polynomial

$$T(x) = 144x^3 - 204x^2 + 37x - 1$$

are β^2 , β_1^2 , and β_2^2 .

Now, transforming $T(x)$ through the relation $y = 4 - 3x$ (that is, substituting $x = (4 - y)/3$ in $T(x)$) leads to $T((4 - y)/3) = -\frac{1}{3}S(y)$, which shows that $\{\alpha^2, \alpha_1^2, \alpha_2^2\} = \{4 - 3\beta^2, 4 - 3\beta_1^2, 4 - 3\beta_2^2\}$. Furthermore, from (1) and (2),

$$\alpha^2 \in (1, \frac{225}{64}), \quad \alpha_1^2 \in (0, 1), \quad \alpha_2^2 \in (\frac{225}{64}, 4),$$

so that $\alpha_1^2 < \alpha^2 < \alpha_2^2$ and

$$4 - 3\beta^2 \in (1, \frac{11}{3}), \quad 4 - 3\beta_1^2 \in (\frac{11}{3}, 4), \quad 4 - 3\beta_2^2 \in (-8, 1),$$

so that $4 - 3\beta_2^2 < 4 - 3\beta^2 < 4 - 3\beta_1^2$. The desired result, $\alpha^2 = 4 - 3\beta^2$, follows.

6. Let f be a function defined on the set of real numbers \mathbb{R} , taking values in \mathbb{R} , and satisfying the condition $f(\cot x) = \sin 2x + \cos 2x$ for every x belonging to the open interval $(0, \pi)$. Find the least and the greatest values of the function $g(x) = f(x) \cdot f(1-x)$ on the closed interval $[-1, 1]$.

Solution by Michel Bataille, Rouen, France.

We show that g has a minimum value of $4 - \sqrt{34}$ and a maximum value of $1/25$ on $[-1, 1]$.

For $x \in (0, \pi)$, we have

$$\begin{aligned} f(\cot x) &= \sin 2x + \cos 2x = 2 \sin x \cos x + \cos^2 x - \sin^2 x \\ &= \sin^2 x (2 \cot x + \cot^2 x - 1) = \frac{\cot^2 x + 2 \cot x - 1}{\cot^2 x + 1}. \end{aligned}$$

Hence, for all $x \in \mathbb{R}$,

$$f(x) = f(\cot(\cot^{-1}(x))) = \frac{x^2 + 2x - 1}{x^2 + 1}.$$

Since $g(\frac{1}{2} + h) = f(\frac{1}{2} + h)f(\frac{1}{2} - h) = g(\frac{1}{2} - h)$ for all real h , it is sufficient to study the values of $g(\frac{1}{2} + h)$ for $h \in [0, \frac{3}{2}]$. An easy computation gives

$$g\left(\frac{1}{2} + h\right) = \frac{16h^4 - 136h^2 + 1}{16h^4 + 24h^2 + 25} = 1 - 8\phi(h^2),$$

where ϕ is defined on $[0, \frac{9}{4}]$ by $\phi(x) = \frac{20x + 3}{16x^2 + 24x + 25}$. Since the derivative $\phi'(x)$ has the same sign as $-80x^2 - 24x + 107$, it follows that ϕ reaches its maximum on $[0, \frac{9}{4}]$ at $x_0 = \frac{\sqrt{34}}{5} - \frac{3}{20}$ with $\phi(x_0) = \frac{20}{32x_0 + 24} = \frac{\sqrt{34} - 3}{8}$ and its minimum at 0 with $\phi(0) = \frac{3}{25}$. Thus, the extreme values of $g(\frac{1}{2} + h)$ are $1 - 8\phi(x_0) = 4 - \sqrt{34}$ (minimum) and $1 - 8\phi(0) = \frac{1}{25}$ (maximum).

7. Let α be a real number, $\alpha \neq 0$. Consider the sequence of real numbers $\{x_n\}$, $n = 1, 2, 3, \dots$, defined by $x_1 = 0$ and $x_{n+1}(x_n + \alpha) = \alpha + 1$ for $n = 1, 2, 3, \dots$.

- (i) Find the general term of the sequence $\{x_n\}$.
- (ii) Prove that the sequence $\{x_n\}$ has a finite limit when $n \rightarrow +\infty$. Find this limit.

Solved by Houda Anoun, Bordeaux, France; and Mohammed Aassila, Strasbourg, France. We give the solution of Anoun, modified by the editor.

For convenience, let $\beta = -\alpha - 1$. Then $\beta \neq -1$, and the given recurrence relation becomes $x_{n+1}(\beta + 1 - x_n) = \beta$.

Let $u_n = \frac{1}{1 - x_n}$ for each n . Then $u_1 = 1$ and for $n = 1, 2, 3, \dots$,

$$\begin{aligned} u_{n+1} &= \frac{1}{1 - x_{n+1}} = \frac{\beta + 1 - x_n}{(\beta + 1 - x_n) - x_{n+1}(\beta + 1 - x_n)} \\ &= \frac{\beta + 1 - x_n}{\beta + 1 - x_n - \beta} = \frac{\beta + 1 - x_n}{1 - x_n} = 1 + \beta u_n. \end{aligned}$$

Case 1. $\beta = 1$.

Then $u_{n+1} = 1 + u_n$. Thus, u_n is an arithmetic sequence, and we have $u_n = n$ for $n = 1, 2, 3, \dots$. Consequently, the general term of the original sequence $\{x_n\}$ is $x_n = 1 - \frac{1}{u_n} = 1 - \frac{1}{n}$, and $\lim_{n \rightarrow \infty} x_n = 1$.

Case 2. $\beta \neq 1$.

Let $v_n = 1 + (\beta - 1)u_n$. Then $v_1 = \beta$ and for $n = 1, 2, 3, \dots$,

$$\begin{aligned} v_{n+1} &= 1 + (\beta - 1)u_{n+1} = 1 + (\beta - 1)(1 + \beta u_n) \\ &= \beta + (\beta - 1)\beta u_n = \beta v_n. \end{aligned}$$

Thus, $\{v_n\}$ is a geometric sequence. For $n = 1, 2, 3, \dots$, we have $v_n = \beta^n$. Then $u_n = \frac{v_n - 1}{\beta - 1} = \frac{\beta^n - 1}{\beta - 1}$, and the general term of the original sequence $\{x_n\}$ is

$$x_n = 1 - \frac{1}{u_n} = 1 - \frac{\beta - 1}{\beta^n - 1} = 1 + \frac{\alpha + 2}{(-\alpha - 1)^n - 1}.$$

If $|\beta| < 1$, then $\lim_{n \rightarrow \infty} \beta^n = 0$, and hence $\lim_{n \rightarrow \infty} x_n = 1 - \frac{\beta - 1}{-1} = \beta$.
If $|\beta| > 1$, then $\lim_{n \rightarrow \infty} |\beta|^n = \infty$, and hence $\lim_{n \rightarrow \infty} x_n = 1$.

In summary, the general term is

$$x_n = \begin{cases} 1 - \frac{1}{n}, & \text{if } \alpha = -2, \\ 1 + \frac{\alpha + 2}{(-\alpha - 1)^n - 1}, & \text{if } \alpha \neq -2, \end{cases}$$

and the sequence $\{x_n\}$ has a finite limit in all cases.

10. For each integer $n > 1$, denote by s_n the number of permutations (a_1, a_2, \dots, a_n) of the first n positive integers such that each permutation satisfies the condition $1 \leq |a_k - k| \leq 2$ for $k = 1, 2, \dots, n$. Prove that $1.75 \cdot s_{n-1} < s_n < 2 \cdot s_{n-1}$ for all integers $n > 6$.

Solution by Mohammed Aassila, Strasbourg, France.

Let n be an integer greater than 6. It can be shown (by induction, for example) that

$$s_n = s_{n-2} + s_{n-4} + 2(s_{n-3} + s_{n-4} + \cdots + s_0).$$

Then, using this to compute s_{n-1} , we see that

$$s_n = s_{n-1} + s_{n-2} + s_{n-3} + s_{n-4} - s_{n-5}. \quad (1)$$

Replacing n by $n - 1$ yields

$$s_{n-1} = s_{n-2} + s_{n-3} + s_{n-4} + s_{n-5} - s_{n-6}. \quad (2)$$

Solving (2) for $s_{n-2} + s_{n-3}$ and substituting in (1), we get

$$s_n = 2s_{n-1} - 2s_{n-5} + s_{n-6}. \quad (3)$$

Since $s_{n-5} > s_{n-6}$, this shows us that $s_n < 2s_{n-1}$, which establishes the right hand inequality.

To prove the left hand inequality, we proceed by induction. It can be directly checked that $s_1 = 0$, $s_2 = 1$, $s_3 = 2$, $s_4 = 4$, and $s_5 = 7$, which means that $\frac{7}{4}s_{n-1} \leq s_n$ for $n = 2, 3, 4$, and 5 , with strict inequality unless $n = 5$. We will assume that $\frac{7}{4}s_{n-1} \leq s_n$ for $n = 2, 3, \dots, k$ for some integer $k \geq 5$, with strict inequality unless $n = 5$. Using this hypothesis four times successively on (3), we get

$$\begin{aligned} s_{k+1} &= 2s_k - 2s_{k-4} + s_{k-5} > 2s_k - 2s_{k-4} \\ &> 2s_k - 2\left(\frac{4}{7}\right)^4 s_k = \frac{4290}{2401} s_k > \frac{7}{4} s_{k-1}, \end{aligned}$$

which establishes the induction.

Next we look at a solution to one of the problems of the XXIX Russian Mathematical Olympiad, V (Final) Round — 10th Form given [2005 : 27–28].

1. (N. Agakhanov) Let M be a set containing 2003 different positive real numbers, such that for any 3 different elements a, b, c from M the number $a^2 + bc$ is rational. Prove that it is possible to choose a natural number n such that for each a from M the number $a\sqrt{n}$ is rational.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.

We present the proof in several steps.

Step 1. If $a, b \in M$ and $a \neq b$, then $a(a + b)$ is rational.

Choose distinct elements $c, d \in M$ different from a and b . Then $a(a + b) = (a^2 + bc) + (d^2 + ab) - (d^2 + bc)$, which is rational.

Step 2. If $a, b \in M$, then b/a is rational.

If $a = b$, then $\frac{b}{a} = 1$, which is clearly rational. On the other hand, if $a \neq b$, then $\frac{b}{a} = \frac{b(a + b)}{a(a + b)}$ is rational by virtue of Step 1.

Step 3. If $a \in M$, then a^2 is rational.

Choose b in M different from a . Then $a(a+b) = a^2 \left(1 + \frac{b}{a}\right)$. Since $a(a+b)$ and $1 + \frac{b}{a}$ are both rational, so is a^2 .

Step 4. If $a \in M$, then there exists a positive integer n and a rational number q such that $a = q\sqrt{n}$.

Since a^2 is rational, we have $a^2 = \frac{r}{s}$, where r and s are positive integers. Then $a = \sqrt{\frac{r}{s}} = \frac{1}{s}\sqrt{rs}$. Take $n = rs$ and $q = \frac{1}{s}$.

Step 5. We now complete the proof.

Fix any element $f \in M$. Then $f = q\sqrt{n}$, where n is a positive integer and q is rational. In view of Step 2, if a is any element of M , then $\frac{a}{f} = q_1$ (rational). Hence, $a = q_1 f = q_1 q \sqrt{n}$. Thus, $a\sqrt{n} = q_1 q n$, which is rational.

And next, the one solution on file from readers for problems of the XXIX Russian Mathematical Olympiad, V Final Round — 11th Form given [2006 : 27–28].

1. (N. Agakhanov, A. Golovanov, V. Senderov) Let α, β, γ , and τ be positive numbers such that, for all x ,

$$\sin \alpha x + \sin \beta x = \sin \gamma x + \sin \tau x.$$

Prove that $\alpha = \gamma$ or $\alpha = \tau$.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.

We need only assume $\alpha + \beta \neq 0$.

Differentiating the given identity three times, we obtain

$$\begin{aligned} \alpha \cos \alpha x + \beta \cos \beta x &= \gamma \cos \gamma x + \tau \cos \tau x, \\ \alpha^3 \cos \alpha x + \beta^3 \cos \beta x &= \gamma^3 \cos \gamma x + \tau^3 \cos \tau x. \end{aligned}$$

In particular, when $x = 0$, we have

$$\begin{aligned} \alpha + \beta &= \gamma + \tau, & (1) \\ \alpha^3 + \beta^3 &= \gamma^3 + \tau^3. & (2) \end{aligned}$$

Cubing both sides of (1), we obtain

$$\alpha^3 + \beta^3 + 3\alpha\beta(\alpha + \beta) = \gamma^3 + \tau^3 + 3\gamma\tau(\gamma + \tau);$$

hence, $\alpha\beta = \gamma\tau$.

Consequently,

$$(\alpha - \gamma)(\alpha - \tau) = \alpha^2 - (\gamma + \tau)\alpha + \gamma\tau = \alpha^2 - (\alpha + \beta)\alpha + \alpha\beta = 0.$$

Therefore, $\alpha = \gamma$ or $\alpha = \tau$.

The next block of solutions from readers are for problems of the Romanian Mathematical Olympiad 9th Grade, given [2006 : 85].

1. Find positive integers a and b such that, for every $x, y \in [a, b]$, we have $\frac{1}{x} + \frac{1}{y} \in [a, b]$.

Solved by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and Matti Lehtinen, National Defence College, Helsinki, Finland. We give the solution by Díaz-Barrero.

Let $x, y \in [a, b]$. From $a \leq x, y \leq b$, we have $\frac{1}{b} \leq \frac{1}{x}, \frac{1}{y} \leq \frac{1}{a}$ and $\frac{2}{b} \leq \frac{1}{x} + \frac{1}{y} \leq \frac{2}{a}$. Since $a \leq \frac{1}{x} + \frac{1}{y} \leq b$, we must have $a \leq \frac{2}{b}$ and $\frac{2}{a} \leq b$, which yields $ab = 2$. Since a and b are integers, the required interval is $[1, 2]$.

2. An integer $n \geq 2$ is called *friendly* if there exists a family A_1, A_2, \dots, A_n of subsets of the set $\{1, 2, \dots, n\}$ such that:

- (i) $i \notin A_i$ for every $i \in \{1, 2, \dots, n\}$;
- (ii) $i \in A_j$ if and only if $j \notin A_i$, for every distinct $i, j \in \{1, 2, \dots, n\}$;
- (iii) $A_i \cap A_j$ is non-empty for every $i, j \in \{1, 2, \dots, n\}$.

Prove: (a) 7 is a friendly number, and (b) n is friendly if and only if $n \geq 7$.

Solution by Matti Lehtinen, National Defence College, Helsinki, Finland.

The table

| | A_1 | A_2 | A_3 | A_4 | A_5 | A_6 | A_7 |
|---|-------|-------|-------|-------|-------|-------|-------|
| 1 | – | – | – | – | + | + | + |
| 2 | + | – | + | – | – | – | + |
| 3 | + | – | – | + | – | + | – |
| 4 | + | + | – | – | + | – | – |
| 5 | – | + | + | – | – | + | – |
| 6 | – | + | – | + | – | – | + |
| 7 | – | – | + | + | + | – | – |

(+ indicates membership, – non-membership in A_i) shows that 7 is friendly. For any $n > 7$, taking A_1, \dots, A_7 from the table and $A_k = \{1, 2, \dots, 7\}$ for $8 \leq k \leq n$, we get a system of sets showing that n is friendly. It remains to show that no n with $2 \leq n \leq 6$ is friendly. Assume, on the contrary, that some $n, 2 \leq n \leq 6$, is friendly and that A_1, \dots, A_n are the subsets involved in the definition of friendliness. Assume that A_1 , say, is the set having the least number of elements among these sets. Assume there is only one element, say 2, in A_1 . By (iii), $A_1 \cap A_2 = \{2\}$, which contradicts (i). This rules out the friendliness of 2. Assume then that A_1 has just two elements, say 2 and 3. Then, by (iii), 2 must be in A_3 and 3 must be in A_2 , in violation

of (ii). This rules out $n = 3$, and shows that every set A_j has to have at least 3 elements. Now (ii) implies that in a membership table like the one above, the number of +'s has to be equal to the number of -'s outside the main diagonal. For $n = 4$, $n = 5$, and $n = 6$ the number of +'s must be 6, 10, and 15, respectively, and these numbers clearly are less than $3n$ in each of the cases.

3. Prove that the mid-points of the altitudes of a triangle are collinear if and only if the triangle is right.

Solution by Matti Lehtinen, National Defence College, Helsinki, Finland.

Consider $\triangle ABC$, with $\angle C = 90^\circ$. Then AC and BC are two of its altitudes. The line connecting their mid-points bisects every line segment connecting C and AB . Now let C be the largest angle in ABC , $C < 90^\circ$. The feet D and E of the altitudes from A and B are on the segments BC and AC . Thus, the distances of the mid-points P and Q of AD and BE from AB is less than the distance of the mid-point S of the altitude CF . Hence, P and Q are in the half-plane determined by the parallel to AB through S . Therefore, P , Q and S are not collinear. Finally, let $C > 90^\circ$. In this case, the feet D and E of the altitudes dropped from A and B lie on the extensions to BC and AC . Their mid-points P and Q now lie farther away from AB than the mid-point S of the altitude CF . Again, P and Q are both in one of the half-planes determined by the parallel to AB through S . Thus, P , Q , and S are not collinear.

4. Let P be a plane. Prove that there exists no function $f : P \rightarrow P$ such that for every convex quadrilateral $ABCD$, the points $f(A)$, $f(B)$, $f(C)$, $f(D)$ are the vertices of a concave quadrilateral.

Solution by Matti Lehtinen, National Defence College, Helsinki, Finland.

Assume such a function f exists. Take any convex pentagon $ABCDE$. Since any four of its five vertices are the vertices of a convex quadrilateral, the convex hull of $\{f(A), \dots, f(E)\}$ has to be a triangle. Assume the vertices are $f(A)$, $f(B)$, $f(C)$. Since no three of the images can be collinear, $f(D)$ and $f(E)$ are distinct interior points of $f(A)f(B)f(C)$. The lines $f(A)f(D)$, $f(B)f(D)$, $f(C)f(D)$ divide $f(A)f(B)f(C)$ into six triangles, and $f(E)$ is an interior point of one of these (again, no three of the five images can be collinear). It is easy to see that $f(E)$, $f(D)$, and some two of the vertices of $f(A)f(B)f(C)$ are vertices of a convex quadrilateral, a contradiction.

That completes this number of the *Corner*. I need your nice solutions and generalizations sent in within a few months, particularly now that the backlog is cleared up and we are looking at using your submissions within a year.

BOOK REVIEWS

John Grant McLoughlin

Math Made Visual

By Claudia Alsina and Roger B. Nelsen, published by the Math Association of America, 2006.

ISBN 0-88385-746-4, hardcover, 190 pages, US\$49.95.

Reviewed by **J. Chris Fisher**, University of Regina, Regina, SK

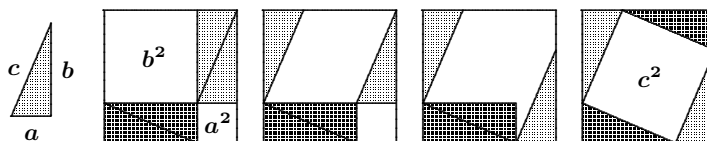
The book begins with a quotation from Martin Gardner:

A dull proof can be supplemented by a geometric analogue so simple and beautiful that the truth of a theorem is almost seen at a glance.

These words sum up the authors' attitude toward proofs by pictures. Their goal here is to provide tools for devising one's own visual proofs. **CRUX with MAYHEM** readers may be familiar with the work of one of the authors, Roger B. Nelsen. In addition to having created many of his own "proofs without words", he has published two collections entitled *Proofs without Words*. Co-author Claudia Alsina likewise has good credentials.

The book comes in three parts. Part I consists of twenty chapters, each five or so pages in length and each describing a method to visualize some mathematical idea. The method is illustrated by means of several examples followed by a handful of exercises called *challenges*. The examples include some of the authors' favorite proofs without words, thankfully now with some words of explanation for those of us who find a proof without any words to be an annoyance.

Some of these proofs without words I have seen before, such as the proof of the Pythagorean Theorem, which comes in the chapter titled "Employing Isometry". You see a square containing four copies of an initial triangle whose sides are labeled a , b , c . The picture makes it quite clear that the total area of the white portion inside the large square remains unchanged as three of the four shaded triangles are translated to new positions. The authors might have included the caption $a^2 + b^2 = c^2$, but were content to leave that to the reader.



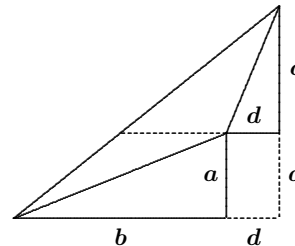
In the same chapter as this proof of the Pythagorean Theorem are four other worked examples with proofs based on rotations and translations. Among the four challenges that follow these results: Find the area of a convex octagon that is inscribed in a circle and has four consecutive sides of length 3 and the remaining four sides of length 2. Happily, Part III of the text consists of solutions or substantial hints to all the challenges of Part I.

(To solve the challenge just mentioned, divide the octagon into isosceles triangles from the centre of the circumcircle. Next rearrange the triangles, alternating the ones with base lengths 2 and 3. This new octagon can be inscribed in a square of side $3 + 2\sqrt{2}$, and whence, the area of the octagon is $(3 + 2\sqrt{2})^2 - 4(\sqrt{2} \cdot \sqrt{2})/2 = 13 + 12\sqrt{2}$.

The authors have done a fine service by collecting together this nice mathematics, thus making it more accessible. I had forgotten about the proof of the median property that appeared in *Mathematics Magazine* in 1990: For positive numbers a, b, c, d ,

$$\frac{a}{b} < \frac{c}{d} \implies \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

The picture shows lines of slope $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{a+c}{b+d}$. It is clear which slopes are steeper. Although the obvious algebraic argument is no harder than this visual argument, the visualization would certainly make a more convincing classroom presentation.



The hundreds of visual arguments included by the authors are all elementary and transparent. However, many of these results have easier and more informative proofs. For example, a theorem of affine geometry affirms that *if a line intersects a hyperbola in points A and B and its asymptotes in A' and B' , then segments AA' and BB' have the same length*. The authors restrict their theorem to a rectangular hyperbola and provide a clumsy verification using coordinates, an approach that seems out of place here. Probably most **CRUX with MAYHEM** readers would introduce affine coordinates with $A' = (1, 0)$, $B' = (0, \pm 1)$, and the origin as the intersection point of the asymptotes, then appeal to symmetry. (This would have been a good place to introduce the notion of an affine reflection, but the authors restrict their transformations to the more familiar Euclidean isometries.) We see in such examples, as well as in a general lack of references to original sources, that the book is simply a collection of items the authors have gathered over the years; it is not intended to be a work of careful scholarship.

For me, the only unsuccessful portion of the book is Part II, where the authors “. . . present some general pedagogical considerations concerning the development of visual thinking, practical approaches for making visualizations in the classroom and, in particular, the role that hands-on materials may play in this process.” The sermon lasts 26 pages, but I saw nothing substantial; if the authors intended an important message, I certainly missed the point. Most of the other 150 pages contain a pleasant variety of interesting theorems, problems, and techniques.

I do not believe that the authors were successful in their goal of teaching readers how to devise their own neat visualizations. I wonder if such a skill can be taught. On the other hand, they have produced a nice collection of results and proofs that are worth knowing. I prefer this book to Nelsen’s previous two collections because of its useful index and its explanations that accompany the visualizations.

Sensational Shape Problems & Other Puzzles

By Ivan Moscovich, published by Sterling Publishing Co., Inc., 2005

ISBN 1-4027-2347-4, soft cover, 128 pages, US\$9.95.

Reviewed by **Tanya Thompson**, Collingwood Collegiate Institute, Collingwood, ON.

This book is one of twelve in Ivan Moscovich's *Mastermind Collection*. It presents recreational mathematics and puzzles visually in a pleasing way that entices the reader to play. As in all his books, the presentation is beautiful, and visual layouts help one to understand the essence of each problem.

This book presents a variety of puzzles (or "Thinkthings", as Moscovich calls them), from historical classics to innovative originals. Dissections, *T*-puzzles, tangrams, Pythagorean Theorem problems, packing puzzles, and geometrical paradoxes are all explored. The book provides answers for the problems, and historical facts are also presented where appropriate.

One of my favourite Thinkthings is a classic from the geometrical paradoxes, called "Disappearing Face Magic", consisting of a line of six faces of men in hats. When the picture is cut into two strips along an indicated black line, and the lower strip is slid to the left, one of the faces disappears. The reader is left to ask, "Which face disappeared?" Martin Gardner, one of the foremost advocates for recreational mathematics, has named this concept the *Principle of Concealed Distribution*. Moscovich mentions this principle in the book. In [1, pp. 117–128], Gardner explains it in greater detail.

As a high school mathematics teacher who loves recreational mathematics, I feel that *Sensational Shape Problems & Other Puzzles* is a great set of engaging problems. These problems are fun, as well as helpful in developing critical thinking and spatial skills necessary for curriculum-based problems. They could be used as warm-up activities or as investigations all their own. A wonderful thing about recreational problems is that they are appropriate for many different levels and abilities. Since basic mathematical skills are not always a requirement, many different learners can find success. With success comes confidence, and for many students confidence is key.

Martin Gardner once wrote, "A teacher of mathematics, no matter how much he loves his subject and how strong his desire to communicate, is perpetually faced with an overwhelming difficulty: How can he keep his students awake? . . . The best way, it has always seemed to me, to make mathematics interesting to students . . . is to approach it in a spirit of play" [2, p. xi]. This book does just that. The Thinkthings motivate students to play. The students will have fun problem-solving and become excited about mathematics. What could be better than that?

References

- [1] Martin Gardner, *Mathematics, Magic and Mystery*, Dover Publications, Inc., New York, 1956.
- [2] Martin Gardner, *1975 Mathematical Carnival: A New Round-up of Tantalizers and Puzzles from "Scientific American"*, Knopf Publishing Group, 1975.

Double Counting Using Incidence Matrices

Yufei Zhao

1. Introduction.

Combinatorics problems appear often on mathematics competitions, and they frequently involve scenarios where individuals are associated with organizations, following a set of rules. Here is one such scenario.

Example 1. In a certain committee, each member belongs to exactly three subcommittees, and each subcommittee has exactly three members. Prove that the number of members is equal to the number of subcommittees.

To investigate problems like this, we need a method of representing and visualizing the setup. We employ incidence matrices for this purpose. In our incidence matrices, each row represents an individual, and each column an organization. A matrix entry is set to 1 if the individual corresponding to its row belongs to the organization corresponding to its column; otherwise, the entry is set to 0. (Of course, the roles of rows and columns could be interchanged.) Two possible incidence matrices for Example 1 are shown.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Let us define the notation that we will be using for incidence matrices. Let r and c denote the number of rows and columns, respectively, let M denote the number of 1s, and let R_i and C_j denote the number of 1s in the i^{th} row and j^{th} column, respectively.

In most of our examples, we will look at an incidence matrix from two perspectives—by rows and by columns. This will allow us to obtain either an identity or an inequality that can be used to prove certain properties.

2. Counting the number of 1s.

When presented with an incidence matrix, one might ask, *how many 1s are there?* That is, what is the value of M ?

If we count the 1s row-by-row, we see that M is the sum of R_i over all rows i . On the other hand, counting the 1s column-by-column yields M as the sum of C_j over all columns j . We have proved the following:

Proposition 1. If A is an incidence matrix with r rows and c columns having row sums R_i , for $i = 1, 2, \dots, r$ and column sums C_j , for $j = 1, 2, \dots, c$, then

$$\sum_{i=1}^r R_i = \sum_{j=1}^c C_j.$$

We now apply this proposition to Example 1, where the incidence matrix has $R_i = C_j = 3$ for all i and j . The equation in the proposition yields $3r = 3c$. Thus, $r = c$, which is the desired result.

3. Counting pairs of 1s.

Often a restriction is imposed that applies to every *pair* of organizations (or individuals). For example, it may be that every two organizations share exactly one common member. Such problems can usually be approached by counting pairs of 1s. Specifically, we are interested in the number of pairs of 1s that lie in the same column (or row).

Proposition 2. Let A be an $r \times c$ incidence matrix with column sums C_j . Suppose that, for every two rows, there exist exactly t columns that contain 1s from both rows. Then

$$t \binom{r}{2} = \sum_{j=1}^c \binom{C_j}{2}.$$

Proof: Let \mathcal{T} denote the set of all unordered pairs of 1s that lie in the same column. We count the elements of \mathcal{T} in two different ways.

Counting by rows: For any two rows, there are t pairs of 1s among these rows that belong to \mathcal{T} ; thus, $|\mathcal{T}| = t \binom{r}{2}$.

Counting by columns: In the j^{th} column, there are C_j 1s and thus $\binom{C_j}{2}$ pairs of 1s. Counting over all the columns gives $|\mathcal{T}| = \sum_{j=1}^c \binom{C_j}{2}$.

The result follows by equating the above two expressions. ■

3.1. Inequalities.

Sometimes we are not given enough information to produce a combinatorial identity. Instead, we have to work with inequalities and bounds.

Many incidence-matrix problems are concerned with the existence of a certain subconfiguration. Such problems are often solved by contradiction. Assuming that the opposite result holds, we can count a particular set (for example, the set of all pairs of 1s that belong to the same column) in two different ways, once by rows and once by columns. If we can establish an upper bound in one count and a lower bound in the other count such that the upper bound is less than the lower bound, then a contradiction is reached.

The above idea is illustrated in the following problem, given in the 2002 International Mathematics Competition for University Students [3].

Example 2. (IMC 2002) Two hundred students participated in a mathematical contest. They had six problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that every problem was solved by at least one of these two students.

Proof: Assume that the contrary is true; that is, for every two students, there is some problem that neither of them solved. Consider the 6×200 incidence matrix for this configuration, where an entry in the matrix is 1 if the student corresponding to the column did *not* solve the problem corresponding to the row, and is 0 otherwise. The setup is illustrated below.

$$\begin{array}{l} \text{Problem 1} \\ \text{Problem 2} \\ \text{Problem 3} \\ \text{Problem 4} \\ \text{Problem 5} \\ \text{Problem 6} \end{array} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$$

Let \mathcal{T} denote the set of pairs of 1s that belong to the same row. We now consider the cardinality of \mathcal{T} from two different perspectives.

Counting by columns: We assumed that for every two students, there was a problem that neither of them solved. Thus, for every two columns, there is at least one pair of 1s among these two columns that belong to the same row. Hence, we can find an element of \mathcal{T} in every pair of columns. Since there are $\binom{200}{2}$ pairs of columns, we have $|\mathcal{T}| \geq \binom{200}{2} = 19,900$.

Counting by rows: We are told that each problem was solved by at least 120 students. This means that there are at most eighty 1s in each row. Thus, each row contains at most $\binom{80}{2}$ pairs of 1s. Since there are six rows, we have $|\mathcal{T}| \leq 6\binom{80}{2} = 18,960$.

The above two inequalities are clearly contradictory. The desired conclusion follows. ■

3.2. Convexity of $\binom{n}{2}$.

Because we are often interested in counting pairs of 1s, the function $f(n) = \binom{n}{2}$ appears frequently. Let us extend this function to the real numbers in the obvious way: $f(x) = \frac{1}{2}x(x-1)$. Note that f is a convex function.

Lemma 1. Let a_1, a_2, \dots, a_n be positive integers, and let $s = \sum_{k=1}^n a_k$. Then

$$\binom{a_1}{2} + \binom{a_2}{2} + \binom{a_3}{2} + \cdots + \binom{a_n}{2} \geq \frac{s(s-n)}{2n}.$$

Proof: Since $f(x) = \frac{1}{2}x(x-1)$ is convex, we have, by Jensen's Inequality,

$$\frac{f(a_1) + f(a_2) + \cdots + f(a_n)}{n} \geq f\left(\frac{s}{n}\right),$$

from which the result follows easily. ■

In fact, we can tighten the bound in Lemma 1.

Lemma 2. Let a_1, a_2, \dots, a_n be positive integers, and let $s = \sum_{k=1}^n a_k$. If $s = nk + r$, where k and r are integers such that $0 \leq r < n$, then

$$\binom{a_1}{2} + \binom{a_2}{2} + \binom{a_3}{2} + \cdots + \binom{a_n}{2} \geq r \binom{k+1}{2} + (n-r) \binom{k}{2}.$$

Proof: Without loss of generality, we may assume that $a_1 \geq a_2 \geq \cdots \geq a_n$. Since a_1, a_2, \dots, a_n are integers, the vector $\langle a_1, a_2, \dots, a_n \rangle$ must majorize the vector $\langle \underbrace{k+1, \dots, k+1}_{r \text{ times}}, \underbrace{k, \dots, k}_{n-r \text{ times}} \rangle$. Since $f(n) = \binom{n}{2}$ is convex, we have, by Karamata's Majorization Inequality [1],

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq rf(k+1) + (n-r)f(k),$$

and the result follows immediately.

We may also use an optimization argument. We want to minimize the value of $\sum_{i=1}^n \binom{a_i}{2}$. Suppose that there exist some indices i and j such that $a_j - a_i > 1$. Then

$$\begin{aligned} & \binom{a_i}{2} + \binom{a_j}{2} - \binom{a_i+1}{2} - \binom{a_j-1}{2} \\ &= \frac{a_i(a_i-1)}{2} + \frac{a_j(a_j-1)}{2} - \frac{a_i(a_i+1)}{2} - \frac{(a_j-1)(a_j-2)}{2} \\ &= a_j - a_i - 1 > 0. \end{aligned}$$

Thus, by replacing any such (a_i, a_j) by (a_i+1, a_j-1) in the sum $\sum_{i=1}^n \binom{a_i}{2}$, we decrease the sum. By repeating this process, we can transform any initial sequence into one where no two terms differ by more than 1. When the process terminates, the sequence must consist of the term $k+1$ repeated r times and the term k repeated $n-r$ times. Since we never increased the sum, the initial sum must be at least as great as the final sum, which is $r \binom{k+1}{2} + (n-r) \binom{k}{2}$. The result follows.

Now we present a problem from the 1998 International Mathematical Olympiad [4] that can be solved using this idea.

Example 3. (IMO 1998) In a competition, there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that $k/a \geq (b-1)/2b$.

Proof: Let us form an incidence matrix as usual. Let there be b rows, each representing a judge, and a columns, each representing a contestant. Make the entries 1 or 0, representing "pass" or "fail", respectively.

Let \mathcal{T} denote the set of pairs of entries that belong to the same column and are either both 0 or 1. Again, we will count \mathcal{T} in two different ways.

Counting by rows: Since the ratings of any two judges coincide for at most k contestants, for every two rows, at most k pairs belong in \mathcal{T} . Since there are $\binom{b}{2}$ ways to choose two rows, we have $|\mathcal{T}| \leq k \binom{b}{2} = \frac{1}{2} kb(b-1)$.

Counting by columns: If a column has p 1s and q 0s, then it contributes $\binom{p}{2} + \binom{q}{2}$ pairs to \mathcal{T} . Note that $p + q = b$ is odd. By Lemma 2,

$$\binom{p}{2} + \binom{q}{2} \geq \binom{\frac{b+1}{2}}{2} + \binom{\frac{b-1}{2}}{2} = \frac{(b-1)^2}{4}.$$

Since there are a columns, we must have $|\mathcal{T}| \geq \frac{1}{4} a(b-1)^2$.

Combining the inequalities for \mathcal{T} , we get $\frac{1}{4} a(b-1)^2 \leq \frac{1}{2} kb(b-1)$. Thus, $k/a \geq (b-1)/2b$. ■

4. Counting with weights.

Let us revisit the idea of counting 1s. However, this time, we will assign a “weight” to each 1 in such a way that the weights of all the 1s in each row sum to 1. Then the sum of all the weights in the matrix is equal to r . The following proposition comes from this idea.

Proposition 3. Let $A = (a_{ij})$ be an $r \times c$ incidence matrix with row sums R_i and column sums C_j .

(a) If $R_i > 0$ for $1 \leq i \leq r$, then $\sum_{i,j} \frac{a_{ij}}{R_i} = r$.

(b) If $C_j > 0$ for $1 \leq j \leq c$, then $\sum_{i,j} \frac{a_{ij}}{C_j} = c$.

Proof: To prove (a), we calculate

$$\sum_{i,j} \frac{a_{ij}}{R_i} = \sum_{i=1}^r \left(\frac{1}{R_i} \sum_{j=1}^c a_{ij} \right) = \sum_{i=1}^r \left(\frac{1}{R_i} R_i \right) = \sum_{i=1}^r 1 = r.$$

The proof of (b) is similar. ■

The following proposition leads to an application of this idea.

Proposition 4. Let $A = (a_{ij})$ be an $r \times c$ incidence matrix with row sums R_i and column sums C_j such that $R_i > 0$ and $C_j > 0$ for $1 \leq i \leq r$ and $1 \leq j \leq c$. If $C_j \geq R_i$ whenever $a_{ij} = 1$, then $r \geq c$.

Proof: If $C_j \geq R_i$ whenever $a_{ij} = 1$, then $\frac{a_{ij}}{R_i} \geq \frac{a_{ij}}{C_j}$ for all $1 \leq i \leq r$ and $1 \leq j \leq c$. From Proposition 3, we have

$$r = \sum_{i,j} \frac{a_{ij}}{R_i} \geq \sum_{i,j} \frac{a_{ij}}{C_j} = c.$$

This completes the proof. ■

Note that $r = c$ if $R_i = C_j$ whenever $a_{ij} = 1$. This equality version of Proposition 4 is somewhat stronger than that used in the solution of Example 1. It is worth noting that when this equality case of Proposition 4 is extended to real matrices, it provides an immediate solution to a problem which appeared recently on the Canadian Mathematical Olympiad [6]. The problem is included below, and the solution is left as an exercise.

Example 4. (CMO 2006) In a rectangular array of non-negative real numbers with m rows and n columns, each row and each column contains at least one positive element. Moreover, if a row and a column intersect in a positive element, then the sums of their elements are the same. Prove that $m = n$.

Now, we will use this technique to solve a problem from the Third Round of the 16th Iranian Mathematical Olympiad 1998–1999 [2].

Example 5. (Iran 1998/1999) Suppose that C_1, \dots, C_n ($n \geq 2$) are circles of radius 1 in the plane such that no two of them are tangent, and the subset of the plane formed by the union of these circles is connected. Let S be the set of points that belong to at least two circles. Show that $|S| \geq n$.

Proof: Let us set up a matrix with n columns, each representing a unit circle, and $|S|$ rows, each representing an intersection point. An entry is 1 if the corresponding point lies on the corresponding circle, and 0 otherwise. Since no circle is disjoint from the rest, and since no two circles are tangent, every column contains at least two 1s. As well, by definition, each row contains at least two 1s. We are required to show that $|S| \geq n$. In light of Proposition 4, we will show that $R_i \leq C_j$ whenever $a_{ij} = 1$.

Suppose that $a_{ij} = 1$. Each 1 in row i distinct from a_{ij} corresponds to a circle that goes through the point represented by row i . Any such circle meets the circle C_j at exactly two points, as no tangency is allowed. We will associate each 1 in row i distinct from a_{ij} with a 1 from column j different from a_{ij} that represents the second intersection. Note that no 1 in column j is associated with two different 1s in row i , as this would mean that three different unit circles are passing through the same two points, which is not possible. Thus, there is an injection from the 1s in row i to the 1s in column j , thereby implying that $R_i \leq C_j$.

$$\left(\begin{array}{ccccccc} & & & \vdots & & & \\ & & & 1 & & & \\ & & & \vdots & & & \\ & & & \vdots & & & \\ \dots & & & a_{ij} = 1 & \dots & 1 & \dots \\ & & & \vdots & & & \\ & & & 1 & \leftarrow & \downarrow & 1 \\ & & & \vdots & & & \end{array} \right)$$

By Proposition 4, the number of rows is greater than or equal to the number of columns, implying that $|S| \geq n$. ■

We will play one more variation on this technique. Sometimes we may not be able to compare R_i and C_j when $a_{ij} = 1$, but we may be able to make the comparison when $a_{ij} = 0$. The next proposition is an analogue of Proposition 4.

Proposition 5. Let $A = (a_{ij})$ be an $r \times c$ incidence matrix with row sums R_i , and column sums C_j , such that $0 < R_i < c$ for $1 \leq i \leq r$ and $0 < C_j < r$ for $1 \leq j \leq c$. If $C_j \geq R_i$ whenever $a_{ij} = 0$, then $r \geq c$.

Proof: Suppose, on the contrary, that $r < c$. Then, whenever $a_{ij} = 0$, we have $0 < r - C_j < c - R_i$, and hence, $\frac{R_i}{c - R_i} < \frac{C_j}{r - C_j}$. Recalling that M denotes the number of 1s in A , we have

$$\begin{aligned} M &= \sum_{i=1}^r R_i = \sum_{i=1}^r (c - R_i) \frac{R_i}{c - R_i} = \sum_{i=1}^r \left(\sum_{j=1}^c (1 - a_{ij}) \right) \frac{R_i}{c - R_i} \\ &= \sum_{i,j} \frac{(1 - a_{ij}) R_i}{c - R_i} < \sum_{i,j} \frac{(1 - a_{ij}) C_j}{r - C_j} \\ &= \sum_{j=1}^c \left(\sum_{i=1}^r (1 - a_{ij}) \right) \frac{C_j}{r - C_j} = \sum_{j=1}^c (r - C_j) \frac{C_j}{r - C_j} = M. \end{aligned}$$

This is clearly impossible. Therefore, $r \geq c$. ■

As an application, the following example is a special case of Fisher's Inequality for block designs [5].

Example 6. Let S_1, S_2, \dots, S_m be distinct subsets of $\{1, 2, \dots, n\}$, such that $|S_i \cap S_j| = 1$ for all $i \neq j$. Prove that $m \leq n$.

Proof: The result holds trivially if the collection is empty ($m = 0$) or if $m = 1$. Thus, we may assume that $m \geq 2$. It is easy to see that none of the sets S_i are empty. Hence, we will assume that all of the sets are non-empty.

As usual, we consider the incidence matrix A for the collection of sets. The m rows of A correspond to sets and the n columns correspond to the elements, where a_{ij} is 1 if element j belongs to set S_i , and 0 otherwise.

Now let us show that the hypotheses of Proposition 5 are satisfied. If any row has all 1s, say the first row, then the constraint $|S_1 \cap S_i| = 1$ for all $i \neq 1$ forces $|S_i| = 1$, which, along with $|S_i \cap S_j| = 1$, implies that $m = 2$, and $n \geq 2$ because the sets are distinct. If any column has all 0s, then that element belongs to none of the sets and we may simply remove that column. We may do this until every column satisfies $C_j \geq 1$ because if the result holds for this reduced matrix, it certainly holds for the original matrix A . Finally, if any column has all 1s, say the first column, then $|S_i \cap S_j| = 1$ implies that no other column may contain two 1s. As well, at most one row may contain a single 1 (in the first column), and each of the other $r - 1$ rows must have the second 1 in distinct columns. Hence, the number of columns must be greater than or equal to the number of rows, giving $m \leq n$ in this case as well. We are now ready to employ Proposition 5.

Let us consider any $a_{ij} = 0$. By the given condition, for every 1 in column j , its corresponding subset must intersect with A_i . Thus, we may associate each 1 on C_j with a 1 in row i such that the element represented by the 1 on R_i also belongs to the subset represented by the 1 on C_j . Note that this association is injective, since having two 1s on C_j both associated with the same 1 in R_i implies that some two subsets intersect in at least two elements. The injective mapping implies that there must be at least as many 1s in the i^{th} row as there are in the j^{th} column.

$$\left(\begin{array}{ccccccc} & & & \vdots & & & \\ & & & 1 & \rightarrow & 1 & \\ & & & \vdots & & \downarrow & \\ \cdots & 1 & \cdots & a_{ij} = 0 & \cdots & 1 & \cdots \\ & \uparrow & & \vdots & & & \\ & 1 & \leftarrow & 1 & & & \\ & & & \vdots & & & \end{array} \right)$$

Therefore, $R_i \geq C_j$ for any $a_{ij} = 0$. It follows from Proposition 5 (with the roles of rows and columns interchanged) that $m \leq n$.

5. Final Remarks.

A large number of combinatorial contest problems can be solved by counting in two ways. Incidence matrices help us to visualize a situation and find the set that should be counted. But it is often easier to bypass the use of a matrix in the final presentation of a solution. The direct use of set theory notation, for example, may give a cleaner presentation at the inconvenience of leaving the reader clueless as to where the ideas came from.

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PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1er septembre 2007**. Une étoile (★) après le numéro indique que le problème a été soumis sans solution.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

We have recently discovered that problem 3188 [2006 : 514, 516] is the same as Mayhem problem M252 [2006 : 264, 265]. We are replacing it in this issue with a new problem. Any solutions for the originally posed problem 3188 will be treated as a solution for Mayhem problem M252. My thanks to Vedula Murty for pointing this out.

3188. Remplacement. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit z_1, z_2, \dots, z_n les zéros du polynôme complexe

$$A(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0,$$

où $a_0 \neq 0$. Montrer que

$$\det \begin{bmatrix} n & z_1 & z_2 & \dots & z_n \\ z_1 & 1 + z_1^2 & 1 & \dots & 1 \\ z_2 & 1 & 1 + z_2^2 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_n & 1 & 1 & \dots & 1 + z_n^2 \end{bmatrix} = a_1^2.$$

3213. *Proposé par Mihály Bencze, Brasov, Roumanie.*

- (a) Soit a et b deux nombres réels positifs avec $a < b$ et soit $f : [a, b] \rightarrow \mathbb{R}$ une fonction continûment différentiable et strictement monotone. Montrer qu'il existe un nombre réel $c \in (a, b)$ tel que

$$(a + b)f(c) = af(a) + bf(b).$$

- (b)★ Soit a_1, a_2, \dots, a_n n nombres réels positifs avec $a_1 < a_2 < \dots < a_n$ et soit $f : [a_1, a_n] \rightarrow \mathbb{R}$ une fonction continûment différentiable et strictement monotone. Montrer qu'il existe un nombre réel $c \in (a_1, a_n)$ tel que

$$\left(\sum_{k=1}^n a_k \right) f(c) = \sum_{k=1}^n a_k f(a_k).$$

3214. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit ABC un triangle acutangle.

- (a) Montrer que $\frac{\tan A}{A} + \frac{\tan B}{B} + \frac{\tan C}{C} > \left(\frac{6}{\pi}\right)^2$.
- (b) Montrer que $A \cot A + B \cot B + C \cot C < \left(\frac{\pi}{2}\right)^2$.
- (c)★ Déterminer les meilleures constantes $0 < c_2 < c_3 \leq \left(\frac{\pi}{2}\right)^2$ et $c_1 \geq \left(\frac{6}{\pi}\right)^2$ telles que

$$\sum_{\text{cyclique}} \frac{\tan A}{A} \geq c_1 \quad \text{et} \quad c_2 \leq \sum_{\text{cyclique}} A \cot A \leq c_3.$$

3215. *Proposé par Shaun White, étudiant, École Secondaire Vincent Massey, Windsor, ON.*

Soit k, ℓ , et m trois entiers plus grand que 2. On appelle un entier n *expressible* pour (k, ℓ, m) s'il existe des nombres réels positifs a_1, a_2, \dots, a_k tels que $\prod_{i=1}^k a_i = 1$ et

$$\sum_{i=1}^k \left(\sum_{j=1}^{\ell} a_{i+j-1} \right)^m = n,$$

où les indices étant pris modulo k .

Supposons que pour un certain triplet (k, ℓ, m) l'entier 1987 n'est pas expressible, tandis que 2005 l'est. Trouver le triplet ordonné (k, ℓ, m) .

3216. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Si a, b, c et d sont des entiers positifs, montrer que

$$45 \left(\frac{1}{a+b+c+d+1} - \frac{1}{(a+1)(b+1)(c+1)(d+1)} \right) \leq 4 + \sum_{\text{cyclique}} \left[\frac{1}{a+1} + \frac{1}{(a+1)(b+1)} \right].$$

3217. *Proposé par Michel Bataille, Rouen, France.*

Soit $\{L_n\}$ la suite de Lucas définie par $L_0 = 2, L_1 = 1$, et, pour $n \geq 1$, $L_{n+1} = L_n + L_{n-1}$. Montrer qu'on a, pour tous les entiers n non négatifs,

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} \frac{L_{2k}}{2^n} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \frac{L_k}{2^{2k}}.$$

3218. *Proposé par Ovidiu Furdui, étudiant, Western Michigan University, Kalamazoo, MI, É-U.*

Soit n un entier tel que $n \geq 2$. Dans \mathbb{R}^n , soit E l'ensemble des points (x_1, x_2, \dots, x_n) tels que $x_i \geq 0$ pour tous les i et $0 < x_1 + x_2 + \dots + x_n \leq 1$. Calculer l'intégrale sur E de la partie fractionnaire de $\frac{1}{x_1 + x_2 + \dots + x_n}$.

3219. *Proposé par Dan Vetter, Regina, SK.*

A l'approche d'une voiture, un vautour (avec une éducation universitaire !) piqueniquant sur la route va toujours s'envoler dans une direction choisie pour maximiser la distance minimale entre lui et la voiture. Montrer que le rapport de la vitesse de la voiture et celle de l'oiseau est $\sec \theta$, où θ est l'angle entre la trajectoire du vautour et la route.

3220. *Proposé par Marian Tetiva, Bîrlad, Roumanie.*

Soit n un entier positif. Montrer qu'on peut trouver une partition de l'ensemble $\{1^2, 2^2, \dots, n^2\}$ des n premiers carrés parfaits en quatre sous-ensembles, ayant chacun la même somme de leurs éléments, si et seulement si $n = 8k$ ou $n = 8k - 1$, pour un certain entier $k \geq 2$.

3221. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

Soit ABC un triangle de côtés a , b et c opposés respectivement aux angles A , B et C . Soit AH la perpendiculaire au côté BC avec H sur BC . Posons $m = BH$ et $n = CH$. Montrer que $a(bm + cn) - bc(b + c)$ est positif, négatif ou nul, suivant que l'angle A est obtus, aigu ou droit.

3222. *Proposé par Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Etant donné des nombres réels positifs a , b et c tels que $a + b + c = 1$, montrer que

$$\frac{(1-a)(1-b)(1-c)}{(1-a^2)^2 + (1-b^2)^2 + (1-c^2)^2} \leq \frac{1}{8}.$$

3223. *Proposé par Achilleas Pavlos Porfyriadis, étudiant, American College de Thessalonique "Anatolia", Thessalonique, Grèce.*

Soit a , b et c des nombres réels positifs satisfaisant

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{abc}.$$

Montrer que

$$\frac{a}{a^2 + 1} + \frac{b}{b^2 + 1} + \frac{c}{c^2 + 1} \leq \frac{3\sqrt{3}}{4}.$$

3224. *Proposé par J. Chris Fisher et Harley Weston, Université de Regina, Regina, SK.*

Soit $A_0B_0C_0$ un triangle isocèle dont l'angle au sommet A_0 est différent de 120° . On définit une suite de triangles $A_nB_nC_n$ dans laquelle le triangle $A_{i+1}B_{i+1}C_{i+1}$ est obtenu à partir du triangle $A_iB_iC_i$ en prenant la réflexion de chaque sommet par rapport au côté opposé (c.-à-d. B_iC_i est perpendiculaire au segment A_iA_{i+1} et le coupe en son milieu). Montrer que les trois angles tendent vers 60° lorsque $n \rightarrow \infty$.

[*Ed* : Ce problème est un cas particulier d'un problème ouvert décrit par Judah Schwartz, "Can technology help us make the mathematics curriculum intellectually stimulating and socially responsible?", *International Journal of Computers for Mathematical Learning*, 4 (1999), pp. 99–119.]

3225★. *Proposé par Panos E. Tsaoussoglou, Athènes, Grèce.*

Soit donné (B, β) un secteur angulaire saillant de sommet B et d'angle β . Soit (A, α) un secteur angulaire saillant dont les côtés ℓ et m coupent ceux de (B, β) en quatre points distincts P, Q, R et S , de sorte que P soit situé entre A et Q d'une part et entre B et S d'autre part.

- (a) Si l'angle α est donné, peut-on choisir A et ℓ de telle sorte que

$$[PBQ] + [APS] = [PQRS],$$

où $[XYZ]$ dénote l'aire du polygone XYZ ?

- (b) Quand peut-on construire les droites ℓ et m avec des outils euclidiens de façon à satisfaire la condition (a) pour une valeur de α donnée?

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3188. *Replacement. Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let z_1, z_2, \dots, z_n be the zeroes of the complex polynomial

$$A(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0,$$

where $a_0 \neq 0$. Prove that

$$\det \begin{bmatrix} n & z_1 & z_2 & \dots & z_n \\ z_1 & 1 + z_1^2 & 1 & \dots & 1 \\ z_2 & 1 & 1 + z_2^2 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_n & 1 & 1 & \dots & 1 + z_n^2 \end{bmatrix} = a_1^2.$$

3213. Proposed by Mihály Bencze, Brasov, Romania.

- (a) Let a and b be positive real numbers with $a < b$ and let $f : [a, b] \rightarrow \mathbb{R}$ be a continuously differentiable and strictly monotone function. Show that there is a real number $c \in (a, b)$ such that

$$(a + b)f(c) = af(a) + bf(b).$$

- (b)★ Let a_1, a_2, \dots, a_n be positive real numbers with $a_1 < a_2 < \dots < a_n$ and let $f : [a_1, a_n] \rightarrow \mathbb{R}$ be a continuously differentiable and strictly monotone function. Show that there is a real number $c \in (a_1, a_n)$ such that

$$\left(\sum_{k=1}^n a_k \right) f(c) = \sum_{k=1}^n a_k f(a_k).$$

3214. Proposed by Mihály Bencze, Brasov, Romania.

Let ABC be an acute-angled triangle.

- (a) Prove that $\frac{\tan A}{A} + \frac{\tan B}{B} + \frac{\tan C}{C} > \left(\frac{6}{\pi}\right)^2$.
- (b) Prove that $A \cot A + B \cot B + C \cot C < \left(\frac{\pi}{2}\right)^2$.
- (c)★ Determine the best constants $c_1 \geq \left(\frac{6}{\pi}\right)^2$ and $0 < c_2 < c_3 \leq \left(\frac{\pi}{2}\right)^2$ such that

$$\sum_{\text{cyclic}} \frac{\tan A}{A} \geq c_1 \quad \text{and} \quad c_2 \leq \sum_{\text{cyclic}} A \cot A \leq c_3.$$

3215. Proposed by Shaun White, student, Vincent Massey Secondary School, Windsor, ON.

Given any integers k, ℓ, m , greater than 2, an integer n is called *expressible* for (k, ℓ, m) if there exist positive real numbers a_1, a_2, \dots, a_k such that $\prod_{i=1}^k a_i = 1$ and

$$\sum_{i=1}^k \left(\sum_{j=1}^{\ell} a_{i+j-1} \right)^m = n,$$

where the subscripts are taken modulo k .

Suppose that for some (k, ℓ, m) the integer 2005 is expressible while 1987 is not. Find the ordered triple (k, ℓ, m) .

3216. Proposed by Mihály Bencze, Brasov, Romania.

If $a, b, c,$ and d are positive integers, prove that

$$45 \left(\frac{1}{a+b+c+d+1} - \frac{1}{(a+1)(b+1)(c+1)(d+1)} \right) \leq 4 + \sum_{\text{cyclic}} \left[\frac{1}{a+1} + \frac{1}{(a+1)(b+1)} \right].$$

3217. Proposed by Michel Bataille, Rouen, France.

Let $\{L_n\}$ be the Lucas sequence defined by $L_0 = 2, L_1 = 1,$ and $L_{n+1} = L_n + L_{n-1}$ for $n \geq 1.$ Prove that, for all non-negative integers $n,$ we have

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} \frac{L_{2k}}{2^n} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \frac{L_k}{2^{2k}}.$$

3218. Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.

Let n be an integer with $n \geq 2.$ In $\mathbb{R}^n,$ let E be the set of points (x_1, x_2, \dots, x_n) such that $x_i \geq 0$ for all i and $0 < x_1 + x_2 + \dots + x_n \leq 1.$ Calculate the integral over E of the fractional part of $\frac{1}{x_1 + x_2 + \dots + x_n}.$

3219. Proposed by Dan Vetter, Regina, SK.

A vulture with a university education, when approached by a car while dining on the road, will always fly off in a direction chosen to maximize the distance of closest approach of the car. Show that the ratio of the speed of the car to the speed of the bird is $\sec \theta,$ where θ is the angle that the vulture's flight path makes with the road.

3220. Proposed by Marian Tetiva, Bîrlad, Romania.

Let n be a positive integer. Prove that the set $\{1^2, 2^2, \dots, n^2\}$ of the first n perfect squares can be partitioned into four subsets each having the same sum of elements if and only if $n = 8k$ or $n = 8k - 1$ for some integer $k \geq 2.$

3221. Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.

Let ABC be a triangle with sides a, b, c opposite the angles $A, B, C,$ respectively. Let AH be perpendicular to the side BC with H on $BC.$ Set $m = BH$ and $n = CH.$ Prove that $a(bm + cn) - bc(b + c)$ is positive, negative, or zero according as $\angle A$ is obtuse, acute, or right-angled.

3222. Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.

Given positive real numbers a, b, c such that $a + b + c = 1$, prove that

$$\frac{(1-a)(1-b)(1-c)}{(1-a^2)^2 + (1-b^2)^2 + (1-c^2)^2} \leq \frac{1}{8}.$$

3223. Proposed by Achilleas Pavlos Porfyriadis, student, American College of Thessaloniki "Anatolia", Thessaloniki, Greece.

Let a, b, c be positive real numbers which satisfy

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{abc}.$$

Prove that

$$\frac{a}{a^2+1} + \frac{b}{b^2+1} + \frac{c}{c^2+1} \leq \frac{3\sqrt{3}}{4}.$$

3224. Proposed by J. Chris Fisher and Harley Weston, University of Regina, Regina, SK.

Let $A_0B_0C_0$ be an isosceles triangle whose apex angle A_0 is not 120° . We define a sequence of triangles $A_nB_nC_n$ in which $\triangle A_{i+1}B_{i+1}C_{i+1}$ is obtained from $\triangle A_iB_iC_i$ by reflecting each vertex in the opposite side (that is, B_iC_i is the perpendicular bisector of A_iA_{i+1} , and so forth). Prove that all three angles approach 60° as $n \rightarrow \infty$.

[Ed: This problem is a special case of an open problem described by Judah Schwartz in "Can technology help us make the mathematics curriculum intellectually stimulating and socially responsible?", *International Journal of Computers for Mathematical Learning*, 4 (1999), pp. 99–119.]

3225★. Proposed by Panos E. Tsaoussoglou, Athens, Greece.

The sides ℓ and m of an acute angle α with vertex A intersect the sides of a fixed acute angle β with vertex B in four distinct points $P, Q, R,$ and S , labelled so that P lies between A and Q and also between B and S .

(a) If the measure of $\angle\alpha$ is fixed, can A and ℓ be chosen so that

$$[PBQ] + [APS] = [PQRS],$$

where $[XYZ]$ denotes the area of polygon XYZ ?

(b) When are the lines ℓ and m constructible with Euclidean tools to satisfy the condition in part (a) for a given fixed value of α ?

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Tom Leong, Brooklyn, NY, USA has indicated that we omitted a crucial line in his featured solution to problem 3098 [2006 : 532-535]. At the end of the paragraph containing equations (1) and (2), it should have been noted that a solution to the problem follows because $s_k(a_m) = -s_k(b_{n-m-k+2})$, and consequently

$$\sum_{K \in \mathcal{F}} P(K) = \sum_{m=1}^{n-k+1} s_k(a_m) + \sum_{m=1}^{n-k+1} s_k(b_m) = 0.$$

We apologize for this omission.

2969. [2004 : 368, 371; 2005 : 411-413] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a, b, c, d , and r be positive real numbers such that $r = \sqrt[4]{abcd} \geq 1$. Prove that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \geq \frac{4}{(1+r)^2}.$$

Remarks by Fan Zhang, Ottawa, ON.

The featured solution to this problem [2005 : 411-413] established the following generalization. For any natural $n > 2$, let $a_1, a_2, \dots, a_n > 0$ such that $a_1 a_2 \cdots a_n = r^n$. Then

$$\frac{1}{(1+a_1)^2} + \frac{1}{(1+a_2)^2} + \cdots + \frac{1}{(1+a_n)^2} \geq \frac{n}{(1 + \sqrt[n]{a_1 a_2 \cdots a_n})^2}$$

if and only if $r \geq \sqrt{n} - 1$.

For $n = 2$, the editor proved that $r \geq \sqrt{2} - 1$ is necessary but not sufficient, and provided a sufficient condition $r \geq 0.5$. The editor pointed out that the minimum sufficient value of r was not known. We will now show that 0.5 is the minimum sufficient value for r . We will do this by showing that $r \geq 0.5$ is necessary in order that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} \geq \frac{2}{(1+r)^2}$$

when a and b are positive real numbers and $r^2 = ab$.

Proposition. Suppose that $\sqrt{2} - 1 < r < 0.5$. Then there exist positive real numbers a and b such that $ab = r^2$ and

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} < \frac{2}{(1+r)^2}.$$

In fact, one may take $a = \frac{1-r^3-r^2-r}{r^2+2r-1}$ and $b = r^2/a$.

Proof: We first observe that, if $a = \frac{1-r^3-r^2-r}{r^2+2r-1}$, then

$$a = \frac{(1-2r+r^4)/(1-r)}{(r+1+\sqrt{2})(r+1-\sqrt{2})} > \frac{r^4/(1-r)}{(r+1+\sqrt{2})(r+1-\sqrt{2})} > 0.$$

To show that $\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} < \frac{2}{(1+r)^2}$, we will prove the equivalent statement, $f(r) < 0$, where

$$f(r) = \left(\frac{1}{(1+a)^2} + \frac{1}{\left(1 + \frac{r^2}{a}\right)^2} - \frac{2}{(1+r)^2} \right) (1+a)^2 \left(1 + \frac{r^2}{a}\right)^2 (1+r)^2 a^2.$$

Using a computer algebra system, $f(r)$ simplifies to

$$\begin{aligned} f(r) &= -2ar^2 + 4a^2r - 6a^2r^2 + 4ar^3 - 2ar^4 + 4a^3r - 2a^3r^2 \\ &\quad + a^4r^2 - 2a^2r^4 - r^4 + 2r^5 + r^6 - 2a^3 - a^4 \\ &= (r-a)^2(2a^2r + a^2r^2 - a^2 + 2ar - 2a \\ &\quad + 2a^4r + 2ar^3 + 2ar^2 - r^2 + r^4 + 2r^3) \\ &= (r-a)^2[(r^2+2r-1)a^2 - 2(1-r^3-r^2-r)a \\ &\quad + (r^2+2r-1)r^2]. \end{aligned}$$

Substituting $a = \frac{1-r^3-r^2-r}{r^2+2r-1}$ and noticing that

$$r-a = \frac{2r^3+3r^2-1}{r^2+2r-1} = \frac{(2r-1)(r+1)^2}{r^2+2r-1},$$

we see that

$$\begin{aligned} f(r) &= \left(\frac{(2r-1)(r+1)^2}{r^2+2r-1} \right)^2 (r^2+2r-1) \left[r^2 - \left(\frac{1-r^3-r^2-r}{r^2+2r-1} \right)^2 \right] \\ &= \frac{(2r-1)^2(r+1)^4[(2r-1)(r+1)^2(r-1)^2]}{(r^2+2r-1)^3} < 0. \end{aligned}$$

3114. [2006 : 107, 109] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Let a, b, c be positive real numbers such that

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 2.$$

Prove that

$$\frac{1}{4a+1} + \frac{1}{4b+1} + \frac{1}{4c+1} \geq 1.$$

I. Solution by Alex Remorov, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON.

Note that for any positive real number x ,

$$\frac{1}{4x+1} \geq \frac{1}{x+1} - \frac{1}{3}, \quad (1)$$

because this inequality is equivalent in succession to

$$\begin{aligned} \frac{1}{4x+1} &\geq \frac{2-x}{3(x+1)}, \\ 3x+3 &\geq (2-x)(4x+1), \\ 4x^2-4x+1 &\geq 0, \\ (2x-1)^2 &\geq 0, \end{aligned}$$

which is obviously true.

Setting $x = a, b, c$ in (1) and adding, we obtain

$$\frac{1}{4a+1} + \frac{1}{4b+1} + \frac{1}{4c+1} \geq \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} - 1 = 1.$$

II. Generalization by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.

We shall prove the generalization that if x_0, x_1, \dots, x_n are positive real numbers such that $\sum_{k=0}^n \frac{1}{x_k+1} = n$, then $\sum_{k=0}^n \frac{1}{n^2x_k+1} \geq 1$. The proposed inequality is the special case where $n = 2$.

Since the result is obvious when $n = 1$, we assume $n > 1$. For any real number x ,

$$\begin{aligned} (n+1)^2(x+1) - (n+1)^2(n^2x+1) + (n^2-1)(n^2x+1)(x+1) \\ = (n+1)^2(x-n^2x) + (n^2-1)(n^2x+1)(x+1) \\ = (n^2-1)((n^2x+1)(x+1) - (n+1)^2x) \\ = (n^2-1)(n^2x^2-2nx+1) = (n^2-1)(nx-1)^2 \geq 0. \end{aligned}$$

with equality if and only if $x = 1/n$. In particular, for positive x , we may divide by $(n+1)^2(x+1)(n^2x+1)$ and re-arrange terms to get

$$\frac{1}{n^2x+1} \geq \frac{1}{x+1} - \frac{n^2-1}{(n+1)^2}, \quad (2)$$

with equality if and only if $x = 1/n$.

Finally, apply (2) to x_0, x_1, \dots, x_n to obtain

$$\sum_{k=0}^n \frac{1}{n^2 x_k + 1} \geq \sum_{k=0}^n \frac{1}{x_k + 1} - (n+1) \frac{n^2 - 1}{(n+1)^2} = n - \frac{n^2 - 1}{n+1} = 1,$$

with equality if and only if $x_0 = x_1 = \dots = x_n = 1/n$.

Also solved by ARKADY ALT, San Jose, CA, USA; ROY BARBARA, University of Beirut, Beirut, Lebanon; MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; JIM BLACK, student, Missouri State University, Springfield, MO, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; DRAGOLJUB MILOŠEVIĆ and G. MILANOVAC, Serbia; VEDULA N. MURTY, Dover, PA, USA; CAO MINH QUANG, Nguyen Binh Khiem specialized high school, Vinh Long, Vietnam; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

About half of the solvers used calculus or convexity and Jensen's Inequality. Zhou showed that the result is actually true for $a, b, c \in (-\infty, -1) \cup (-1/4, \infty)$. Several other generalizations were obtained. Benito, Ciaurri, and Fernández proved that if $n \geq 3$ and a_1, \dots, a_n are positive real numbers such that $\sum_{i=1}^n \frac{1}{a_i+1} = 2$, then $\sum_{i=1}^n \frac{1}{k^2 a_i+1} \geq 1$, for $k = \frac{n+1}{n-1}$. Their proof is a straightforward generalization of Solution 1 above. Janous proved that if $n \geq 2$ and x_1, x_2, \dots, x_n are positive real numbers such that $\sum_{i=1}^n \frac{1}{x_i+1} = a$, where $a < n$ is a constant, then $\sum_{i=1}^n \frac{1}{bx_i+1} \geq \frac{an}{b(n-a)+a}$ for all constants $b > 1$. The special case when $n = 3$, $a = 2$, and $b = 4$ is the proposed inequality. Quang established the similar result that if $\sum_{i=1}^n \frac{1}{x_i+1} \geq 1$, then $\sum_{i=1}^n \frac{1}{4x_i+1} \geq \frac{n}{4n-3}$.

3115. [2006 : 107, 109] Proposed by Arkady Alt, San Jose, CA, USA.

Let a, b, c , be the lengths of the sides opposite the vertices A, B, C , respectively, in triangle ABC . Prove that

$$\frac{\cos^3 A}{a} + \frac{\cos^3 B}{b} + \frac{\cos^3 C}{c} < \frac{a^2 + b^2 + c^2}{2abc}.$$

Essentially the same solution by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; and Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let R be the circumradius of $\triangle ABC$. By the Law of Sines, we have

$$\begin{aligned} & \sum_{\text{cyclic}} (b^2 + c^2 - a^2) \sin^2 A \\ &= \sum_{\text{cyclic}} \frac{a^2(b^2 + c^2 - a^2)}{4R^2} = \frac{2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)}{4R^2} \\ &= \frac{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}{4R^2} > 0. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{\text{cyclic}} (b^2 + c^2 - a^2) \cos^2 A &= \sum_{\text{cyclic}} (b^2 + c^2 - a^2)(1 - \sin^2 A) \\ &< \sum_{\text{cyclic}} (b^2 + c^2 - a^2) = a^2 + b^2 + c^2, \end{aligned}$$

which is equivalent to $\sum_{\text{cyclic}} (2bc \cos^3 A) < a^2 + b^2 + c^2$. Dividing both sides by $2abc$, the result follows immediately.

Also solved by MOHAMMED AASSILA, Strasbourg, France; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, University of Beirut, Beirut, Lebanon; MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VEDULA N. MURTY, Dover, PA, USA; ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer. There were also two incorrect solutions.

Both Janous and Zvonaru showed that the given inequality is equivalent to

$$\sum_{\text{cyclic}} a^2(b^2 + c^2 - a^2)^3 < 4a^2b^2c^2(a^2 + b^2 + c^2),$$

and remarked that this is a special case of Crux problem #3116 (by the same proposer). Zvonaru also pointed out that if $\triangle ABC$ is an acute triangle, then the following is a very simple proof of the given inequality:

$$\sum_{\text{cyclic}} \frac{\cos^3 A}{a} < \sum_{\text{cyclic}} \frac{\cos A}{a} = \sum_{\text{cyclic}} \frac{b^2 + c^2 - a^2}{2abc} = \frac{a^2 + b^2 + c^2}{2abc}.$$

3116. [2006 : 107, 110] Proposed by Arkady Alt, San Jose, CA, USA.

For arbitrary real numbers a, b, c , prove that

$$\sum_{\text{cyclic}} a(b + c - a)^3 \leq 4abc(a + b + c).$$

Essentially the same solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Michel Bataille, Rouen, France; and Joel Schlosberg, Bayside, NY, USA.

$$4abc(a + b + c) - \sum_{\text{cyclic}} a(b + c - a)^3 = (a^2 + b^2 + c^2 - 2ab - 2ac - 2bc)^2 \geq 0.$$

The equality holds if and only if $a = b$ and $c = 0$, $b = c$ and $a = 0$, or $c = a$ and $b = 0$.

Also solved by ROY BARBARA, University of Beirut, Beirut, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; VEDULA N. MURTY, Dover, PA, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3117. [2006 : 107, 110] *Proposed by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Let a, b, c be the lengths of the sides and s the semi-perimeter of $\triangle ABC$. Prove that

$$\sum_{\text{cyclic}} (a+b)\sqrt{ab(s-a)(s-b)} \leq 3abc.$$

Essentially the same solution by Mohammed Aassila, Strasbourg, France; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and Titu Zvonaru, Comănești, Romania.

We have

$$\begin{aligned} abc^2 - (a+b)^2(s-a)(s-b) &= abc^2 - (a+b)^2 \left[\frac{c^2 - (a-b)^2}{4} \right] \\ &= \frac{(4ab - (a+b)^2)c^2 + (a+b)^2(a-b)^2}{4} \\ &= \frac{1}{4}(a-b)^2[(a+b)^2 - c^2] \geq 0, \end{aligned}$$

since $c \leq a+b$ by the Triangle Inequality. Equality holds if and only if $a = b$. It follows that $(a+b)\sqrt{ab(s-a)(s-b)} \leq \sqrt{ab}\sqrt{abc^2} = abc$, with equality if and only if $a = b$. Then

$$\sum_{\text{cyclic}} (a+b)\sqrt{ab(s-a)(s-b)} \leq \sum_{\text{cyclic}} abc = 3abc,$$

with equality if and only if $\triangle ABC$ is equilateral.

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN G. HEUVER, Grande Prairie, AB; DRAGOLJUB MILOŠEVIĆ and G. MILANOVAC, Serbia; VEDULA N. MURTY, Dover, PA, USA; ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3118. [2006 : 108, 110] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let BE and CF be altitudes of the acute-angled triangle ABC with E on AC and F on AB . Let BK and CL be the interior angle bisectors of $\angle ABC$ and $\angle ACB$, respectively, with K on AC and L on AB . Let I denote the incentre of $\triangle ABC$, and let O denote its circumcentre. Prove that E, F , and I are collinear if and only if K, L , and O are collinear.

I. Solution by Francisco Bellot Rosado, I. B. Emilio Ferrari, Valladolid, Spain.

This nice problem was proposed (but not used) at the 38th IMO held at Mar del Plata (Argentina), in 1997.

We use a result known as the “Theorem of Transversals”, which establishes necessary and sufficient conditions for a line which cuts two sides of a triangle to pass through some of the noteworthy points of the triangle. For a detailed discussion of the subject, see [1].

Since $\triangle ABC$ is acute, a necessary and sufficient condition for K , L , and O to be collinear is

$$\frac{BL}{LA} \cdot \sin 2B + \frac{CK}{KA} \cdot \sin 2C = \sin 2A. \quad (1)$$

A necessary and sufficient condition for E , F , and I to be collinear is

$$\frac{BF}{FA} \cdot b + \frac{CE}{EA} \cdot c = a. \quad (2)$$

In our case, since BE and CF are altitudes, we have $BF = a \cdot \cos B$, $FA = b \cdot \cos A$, $CE = a \cdot \cos C$, and $EA = c \cdot \cos A$. Thus, equation (2) takes the form $\frac{\cos B}{\cos A} + \frac{\cos C}{\cos A} = 1$; that is,

$$\cos B + \cos C = \cos A. \quad (3)$$

On the other hand, since BK and CL are internal bisectors, the theorem on internal bisectors gives us directly that $\frac{BL}{LA} = \frac{a}{b}$ and $\frac{CK}{KA} = \frac{a}{c}$. Therefore, equation (1) can be written as $\frac{a}{b} \sin 2B + \frac{a}{c} \sin 2C = \sin 2A$; that is,

$$\cos B \left(\frac{\sin B}{b} \right) + \cos C \left(\frac{\sin C}{c} \right) = \cos A \left(\frac{\sin A}{a} \right).$$

Using the Law of Sines, we reduce this equation to (3), and we are done.

II. Solution by Michel Bataille, Rouen, France.

In areal coordinates relative to $\triangle ABC$, we have

$$I(a, b, c), \quad E(a \cos C, 0, c \cos A), \quad \text{and} \quad F(a \cos B, b \cos A, 0).$$

Hence, I , E , and F are collinear if and only if

$$\begin{vmatrix} a & a \cos C & a \cos B \\ b & 0 & b \cos A \\ c & c \cos A & 0 \end{vmatrix} = 0,$$

which reduces to $\cos A(\cos B + \cos C - \cos A) = 0$.

Similarly, from the areal coordinates $O(a \cos A, b \cos B, c \cos C)$, $K(a, 0, c)$, and $L(a, b, 0)$, we see that O , K , and L are collinear if and only if $\cos B + \cos C - \cos A = 0$.

In conclusion, if O , K , and L are collinear, then I , E , and F are also collinear (in any triangle ABC). Conversely, if I , E , and F are collinear and $\angle A \neq 90^\circ$, then O , K , and L are collinear.

References

- [1] Francisco Bellot Rosado, *Un théorème peu connu : le théorème des transversales*; *Mathématique et Pédagogie*, n° 153, 41–55, 2005.

Also solved by MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Demis posed the following question: If the side BC is a constant and the points K , L , and O remain collinear, then

1. does A lie on an interesting curve?
2. do we obtain the maximum or minimum distance of A from BC if and only if $AB = AC$?

3119. [2006 : 108, 110] Proposed by Michel Bataille, Rouen, France.

Let r and s denote the inradius and semi-perimeter, respectively, of triangle ABC . Show that

$$3\sqrt{3}\sqrt{\frac{r}{s}} \leq \sqrt{\tan\left(\frac{1}{2}A\right)} + \sqrt{\tan\left(\frac{1}{2}B\right)} + \sqrt{\tan\left(\frac{1}{2}C\right)} \leq \sqrt{\frac{s}{r}}.$$

A combination of nearly identical solutions from Mohammed Aassila, Strasbourg, France; Li Zhou, Polk Community College, Winter Haven, FL, USA; and the proposer.

Since $\tan\frac{A}{2} = \frac{r}{s-a}$, $\tan\frac{B}{2} = \frac{r}{s-b}$, and $\tan\frac{C}{2} = \frac{r}{s-c}$, the proposed inequalities may be rewritten as

$$\frac{3\sqrt{3}}{\sqrt{s}} \leq \frac{1}{\sqrt{s-a}} + \frac{1}{\sqrt{s-b}} + \frac{1}{\sqrt{s-c}} \leq \frac{\sqrt{s}}{r}.$$

The inequality on the left is an immediate consequence of Jensen's Inequality (which, in the present context, is just the Power Mean Inequality). Specifically, using the convexity of $1/\sqrt{t}$, we obtain

$$\frac{1}{\sqrt{s-a}} + \frac{1}{\sqrt{s-b}} + \frac{1}{\sqrt{s-c}} \geq 3 \cdot \frac{1}{\sqrt{\frac{(s-a) + (s-b) + (s-c)}{3}}} = \frac{3\sqrt{3}}{\sqrt{s}}.$$

Equality holds if and only if $\triangle ABC$ is equilateral.

The other inequality is a consequence of the AM-GM Inequality, as follows:

$$\begin{aligned} & \sqrt{(s-a)(s-b)(s-c)} \left(\frac{1}{\sqrt{s-a}} + \frac{1}{\sqrt{s-b}} + \frac{1}{\sqrt{s-c}} \right) \\ &= \sqrt{(s-b)(s-c)} + \sqrt{(s-c)(s-a)} + \sqrt{(s-a)(s-b)} \\ &\leq \frac{2s-b-c}{2} + \frac{2s-c-a}{2} + \frac{2s-a-b}{2} \\ &= s = \frac{\text{Area}(ABC)}{r} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{r}, \end{aligned}$$

which yields the desired result. Once again, equality holds if and only if $\triangle ABC$ is equilateral.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ÓSCAR CIAURRI, Universidad de La Rioja, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D.M. MILOŠEVIĆ, Pranjani, Yugoslavia; VEDULA N. MURTY, Dover, PA, USA; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and TITU ZVONARU, Comănești, Romania.

3120. [2006 : 108, 110] Proposed by Michel Bataille, Rouen, France.

Let ABC be an isosceles triangle with $AB = BC$, and let F be the mid-point of AC . Let $\alpha = \angle BAX$, where X is a variable point on the ray BF . As long as $\alpha \neq \pi/2$, the reflections of the line BF in BA and XA intersect. Let that point of intersection be denoted by M .

Find $\lim_{\alpha \rightarrow \pi/2} |\cos \alpha| \cdot CM$.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

Let $r = AF (= \frac{1}{2}AC)$. We shall see that r is the desired limit. Since line MB is the reflection of BF across BA , and line MX is the reflection of line BF across AX , then A is equidistant from the three sides of $\triangle BMX$. Indeed, if $\alpha (= \angle BAX)$ is sufficiently near $\pi/2$, then r is the inradius when A and M are on the same side of BX , and the exradius otherwise. Let $\theta = \angle BMA = \angle XMA$. Then $\alpha - \theta = \pi/2$ when A is the incentre, and $\alpha + \theta = \pi/2$ when A is the excentre. Either way, we have $|\cos \alpha| = |\sin \theta|$. Next, observe that the line MB is fixed, and as $\alpha \rightarrow \pi/2$, the point M goes to infinity along that fixed line and $MA/MC \rightarrow 1$. But $|\cos \alpha| = |\sin \theta|$; therefore,

$$\lim_{\alpha \rightarrow \pi/2} |\cos \alpha| MC = \lim_{\alpha \rightarrow \pi/2} |\sin \theta| MA = r = AF.$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, University of Beirut, Beirut, Lebanon; ÓSCAR CIAURRI, Universidad de La Rioja, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA (two solutions); JOEL SCHLOSBERG, Bayside, NY, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

3121. [2006 : 108, 110] Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

Let n and r be positive integers. Show that

$$\left(\frac{1}{2^n} \sum_{k=1}^n \frac{1}{k} \binom{n-1}{k-1} \left[1 - \frac{1}{2^{nr}} \binom{n}{k}^r \right] \right)^r \leq \frac{r^r}{(r+1)^{r+1}}.$$

Solution by Michel Bataille, Rouen, France.

The proposed inequality is equivalent to

$$\frac{1}{2^n} \sum_{k=1}^n \frac{1}{k} \binom{n-1}{k-1} \left[1 - \frac{1}{2^{nr}} \binom{n}{k}^r \right] \leq \frac{r}{r+1} \cdot \frac{1}{(r+1)^{1/r}}. \quad (1)$$

Let L denote the left side of (1). Then

$$L = \frac{1}{n} \sum_{k=1}^n \frac{1}{2^n} \binom{n}{k} \left[1 - \left(\frac{1}{2^n} \binom{n}{k} \right)^r \right].$$

Since $\sum_{k=0}^n \binom{n}{k} = 2^n$, we have $0 < \frac{1}{2^n} \binom{n}{k} < 1$ for $k = 1, 2, \dots, n$.

Using simple calculus, it is easy to show that the function f defined by $f(x) = x(1-x^r)$, for $x \in [0, 1]$, attains its absolute maximum M at $x = \frac{1}{(r+1)^{1/r}}$. Hence,

$$\begin{aligned} L &= \frac{1}{n} \sum_{k=1}^n f\left(\frac{1}{2^n} \binom{n}{k}\right) \leq \frac{1}{n} \sum_{k=1}^n M = M \\ &= f\left(\frac{1}{(r+1)^{1/r}}\right) = \frac{r}{r+1} \cdot \frac{1}{(r+1)^{1/r}}, \end{aligned}$$

and (1) follows.

Also solved by MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; WÄLTER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

The proof featured above actually shows that the given inequality holds for all positive real numbers r . This was pointed out explicitly by Benito, Ciaurri, and Fernández.

3122. [2006 : 108, 111] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let $\triangle ABC$ and $\triangle A'B'C'$ have right angles at A and A' , respectively, and let h_a and $h_{a'}$ denote the altitudes to the sides a and a' , respectively. If $b \geq c$ and $b' \geq c'$, prove that

$$\sqrt{aa'} + 2\sqrt{h_a h_{a'}} \leq \sqrt{2}(\sqrt{bb'} + \sqrt{cc'}).$$

Solution by Michel Bataille, Rouen, France.

Since $ah_a = bc$ and $a'h_{a'} = b'c'$, the proposed inequality may be expressed as

$$\sqrt{aa'} + 2\sqrt{\frac{bb'cc'}{aa'}} \leq \sqrt{2}(\sqrt{bb'} + \sqrt{cc'}).$$

Squaring and multiplying by aa' gives the equivalent inequality

$$(aa')^2 + 4bb'cc' \leq 2aa'(bb' + cc'),$$

which may be rewritten as

$$(aa' - 2bb')(aa' - 2cc') \leq 0. \quad (1)$$

By means of the Cauchy-Schwarz Inequality, we get

$$aa' = \sqrt{b^2 + c^2} \sqrt{(b')^2 + (c')^2} \geq bb' + cc' \geq 2cc'$$

(the last inequality because $b \geq c$ and $b' \geq c'$). Thus, the second factor of (1) is non-negative.

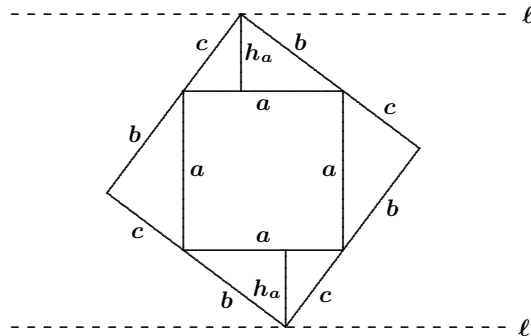
Since $b^2 + c^2 \leq 2b^2$ and $(b')^2 + (c')^2 \leq 2(b')^2$, we get

$$aa' = \sqrt{b^2 + c^2} \sqrt{(b')^2 + (c')^2} \leq \sqrt{2b} \cdot \sqrt{2b'} = 2bb',$$

and the first factor in (1) is non-positive. The result follows at once.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, University of Beirut, Beirut, Lebanon (2 solutions); CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Barbara and the proposer point out that, if the triangles are similar, then the proposed inequality simplifies to $a + 2h_a \leq \sqrt{2}(b + c)$. Barbara provided a visualization of this inequality, shown below. Clearly, $a + 2h_a$ is the distance from ℓ to ℓ' and $\sqrt{2}(b + c)$ is the length of the diagonal of the large square.



3123. [2006 : 111] Proposed by Joe Howard, Portales, NM, USA.

Let a, b, c be the sides of a triangle. Show that

$$\frac{abc(a + b + c)^2}{a^2 + b^2 + c^2} \geq 2abc + \prod_{\text{cyclic}} (b + c - a).$$

Solution by Titu Zvonaru, Comănești, Romania.

We prove that the given inequality is true for any three non-negative numbers a , b , and c such that $a^2 + b^2 + c^2 > 0$.

For any such a , b , and c , we have

$$\begin{aligned}
 & abc(a+b+c)^2 - (a^2 + b^2 + c^2) \left(2abc + \prod_{\text{cyclic}} (b+c-a) \right) \\
 &= abc \left(\sum_{\text{cyclic}} a^2 + 2 \sum_{\text{cyclic}} bc \right) - (a^2 + b^2 + c^2) \left(\sum_{\text{cyclic}} (a^2b + a^2c) - \sum_{\text{cyclic}} a^2 \right) \\
 &= \sum_{\text{cyclic}} (a^3bc + 2ab^2c^2) - \sum_{\text{cyclic}} (a^4b + a^4c + 2ab^2c^2 - a^5) \\
 &= \sum_{\text{cyclic}} a^3(a^2 + bc - ab - ac) = \sum_{\text{cyclic}} a^3(a-b)(a-c) \geq 0,
 \end{aligned}$$

by Schur's Inequality. The desired inequality follows immediately. Equality holds if and only if $a = b = c$ or two of the numbers a, b, c are equal and the third is zero.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DRAGOLJUB MILOŠEVIĆ and G. MILANOVAC, Serbia; VEDULA N. MURTY, Dover, PA, USA; ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

3124. [2006 : 109, 111] *Proposed by Joe Howard, Portales, NM, USA.*

Let a, b, c be the sides of $\triangle ABC$ in which at most one angle exceeds $\pi/3$, and let r be its inradius. Show that

$$\frac{\sqrt{3}(abc)}{a^2 + b^2 + c^2} \geq 2r.$$

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Let R and s denote the circumradius and semiperimeter of $\triangle ABC$, respectively. We first use the well-known formulas $abc = 4Rrs$ and $a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)$ to write the given inequality as $\sqrt{3}Rs \geq s^2 - r^2 - 4Rr$. This is equivalent to

$$s^2 - \sqrt{3}Rs - r(4R + r) \leq 0,$$

or $(s - x_1)(s + x_2) \leq 0$, where

$$x_1 = \frac{\sqrt{3}R + \sqrt{3R^2 + 16Rr + 4r^2}}{2}$$

and

$$x_2 = \frac{-\sqrt{3}R + \sqrt{3R^2 + 16Rr + 4r^2}}{2}.$$

Obviously, $s + x_2 > 0$. Therefore, we just have to prove that $s \leq x_1$.

Now, it is known ([1, section 37, and also p. 256]) that for any triangle satisfying the given condition, we have $s \leq \sqrt{3}(R + r)$. [Ed: A proof of this may also be found in Howard's featured solution to #2887 [2004 : 519].] To show that $s \leq x_1$, it then suffices to show that

$$2\sqrt{3}(R + r) \leq \sqrt{3}R + \sqrt{3R^2 + 16Rr + 4r^2}.$$

Using some simple algebra, it is easily seen that this inequality is equivalent to $3(R + 2r)^2 \leq 3R^2 + 16Rr + 4r^2$, or $2r \leq R$, which is a celebrated and well-known result of Euler.

References

- [1] A. Bager, *A Family of Goniometric Inequalities*, Univ. Beograd. Publ. El. Fak. Ser. Mat. Fiz. 338–352 (1971), 5–25.
- [2] D.S. Mitrinović et al., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989.

Also solved by MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; VEDULA N. MURTY, Dover, PA, USA; and the proposer whose proof is virtually the same as the one given above. There was also an incomplete solution.

Janous mentioned that triangles which satisfy the described condition were “baptized” as “triangles of Bager’s type II” (see [2, 256–261]).

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